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Probabilistic Applications of Tauberian Theorems

A.L. YAKIMIV

TRANSLATED BY ANDREI V. KOLCHIN

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Introduction

By Abelian theorems are meant those assertions which allow to deduce from the asymptotic behaviour of sequences and functions the asymptotic properties of their generating functions and Laplace transforms (as well as other integral transforms). Theorems converse to Abelian are referred to as Tauberian. They are named after Abel and Tauber, respectively, who were the first to prove theorems of such kinds (Abel, 1826; Tauber, 1897)).

Usually, direct methods are used to prove Abelian theorems. It is much more difficult to prove the corresponding Tauberian theorems, and a wide spectrum of analytical techniques is involved. As milestones in Tauberian theory, we mention the works (Littlewood, 1910; Hardy, Littlewood, 1914; Karamata, 1930b; Karamata, 1931a; Wiener, 1932; Korevaar, 1954; Keldysh, 1973; Vladimirov, 1978).

In last three decades, much thought has been given to multidimensional Tauberian theory. This is primarily due to the fact that Tauberian theorems are finding ever-widening application in mathematical physics, theory of differential equations, and probability theory.

We place particular emphasis on the multidimensional studies (Alpár, 1976; Alpár, 1984; Vladimirov, 1978; Vladimirov *et al.*, 1988; Drozhzhinov, 1983; Drozhzhinov, Zavyalov, 1984; Drozhzhinov, Zavyalov, 1986a; Drozhzhinov, Zavyalov, 1986b; Drozhzhinov, Zavyalov, 1990; Drozhzhinov, Zavyalov, 1992; Drozhzhinov, Zavyalov, 1995a; Drozhzhinov, Zavyalov, 1995b; Drozhzhinov, Zavyalov, 1998; Drozhzhinov, Zavyalov, 2000; Drozhzhinov, Zavyalov, 2002; de Haan, Omey, 1983; de Haan *et al.*, 1984; Resnick, 1991; Omey, 1989; Omey, Willekens, 1989; Diamond, 1987; Kozlov, 1983; Pilipović *et al.*, 1990; Stanković, 1983; Stanković, 1985a; Stanković, 1985b; Stadtmüller, Trautner, 1979; Stadtmüller, Trautner, 1981; Stadtmüller, 1983; Stam, 1977; Chelidze, 1977).

We omit the discussion of Tauberian theorems with remainder term; the reader can find the details in the books (Aljančić *et al.*, 1974; Ganelius, 1971; Postnikov, 1980; Subkhankulov, 1966) and the papers (Frennemo, 1965; Frennemo, 1966; Vladimirov *et al.*, 1988; Drozhzhinov, Zavyalov, 1995a; Drozhzhinov, Zavyalov, 1995b). We note, though, that the use of ordinary Tauberian theorems (without a remainder term) has allowed the author to obtain an exact asymptotic expression of the remainder term for infinitely divisible distributions (see Chapter 4).

The Tauberian theory has found a widespread application in probability theory. Tauberian theorems have been used to study asymptotic problems of probability theory by (Vatutin, 1977b; Vatutin, 1977c; Vatutin, 1979; Vatutin, Sagitov, 1988a; Zolotarev, 1961; Novikov, 1982; Postnikov, 1980; Rogozin, 2002a; Rogozin, 2002b; Sevastyanov, 1978; Seneta, 1969; Seneta, 1973; Seneta, 1974; Feller, 1966; Bingham, 1984a; Bingham, 1984b;

Bingham, 1988; Bingham, 1989; Bingham, Doney, 1974; de Haan, Omey, 1983; de Haan *et al.*, 1984; Omey, 1989; Weiner, 1990)), to name a few.

Despite of the strong interest of probabilists to Tauberian theorems, no book specially devoted to this topic has been published yet. This monograph is intended to fill this gap.

A series of studies (Yakymiv, 1981; Yakymiv, 1982; Yakymiv, 1983; Yakymiv, 1984; Yakymiv, 1987a; Yakymiv, 1987b; Yakymiv, 1988; Yakymiv, 1990a; Yakymiv, 1990b; Yakymiv, 1991a; Yakymiv, 1991b; Yakymiv, 1993a; Yakymiv, 1993b; Yakymiv, 1995; Yakymiv, 1997; Yakymiv, 2000; Yakymiv, 2001; Yakymiv, 2002; Yakymiv, 2002; Yakymiv, 2003a; Yakymiv, 2003b), which have seen the light in the last two decades, are also devoted to probabilistic applications of Tauberian theorems and form the basis for this book. It contains Tauberian theorems and their applications to analysing the asymptotic behaviour of stochastic processes, record processes, random permutations, and infinitely divisible random variables. We include the works on branching processes (Vatutin, 1977b; Sevastyanov, 1978) which give the impetus to our studies in this field. We also include a series of Tauberian theorems due to Drozhzhinov and Zavyalov which, we believe, are of much interest to probabilists, although they were intended for use in other fields of mathematics.

The book on Tauberian theorems and their applications follows the traditions of the Steklov Institute of Mathematics. It suffices to mention the books (Postnikov, 1988; Vladimirov *et al.*, 1988) and the papers (Keldysh, 1973; Sevastyanov, 1978; Vatutin, 1977b).

Tauberian theorems are contained in the first chapter of the book. In particular, multidimensional extensions of Tauberian theorems due to Karamata are given in Section 1.3; a multidimensional Tauberian comparison theorem of Keldysh type and an extension of a Tauberian theorem of Littlewood type are given in Section 1.5; Section 1.6 contains a series of one-dimensional Tauberian theorems. The whole Section 1.7 is devoted to Tauberian theorems due to Drozhzhinov and Zavyalov. In Section 1.8, three multidimensional Tauberian theorems of Drozhzhinov–Zavyalov type are given, which are used later to study the asymptotic behaviour of infinitely divisible distributions in a cone. Sections 1.1, 1.2 and 1.4 are of auxiliary nature; here we deal with generalisations of regularly varying functions occurring in Tauberian theorems.

Chapters 2–5 cover probabilistic applications of Tauberian theorems. In Chapter 2, asymptotic properties of branching processes are studied. A series of limit theorems on random permutations whose cycle lengths belong to a given set A (the so-called A-permutations) constitute Chapter 3. Chapter 4 is devoted to analysing the asymptotic behaviour of infinitely divisible distributions at infinity. In Chapter 5, probabilities of large deviations for some random variables are studied in the context of the record model.

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1

Tauberian theorems

1.1. Regularly varying functions in a cone

A Borel set $\Gamma \subseteq \mathbf{R}^n$ is said to be a *cone* with apex at a point $a \in \mathbf{R}^n$ if $a + \lambda(x - a) \in \Gamma$ for any $x \in \Gamma$, $\lambda > 0$, $\lambda \in \mathbf{R}$. Let $\operatorname{int} A$ denote the interiority of the set $A \subseteq \mathbf{R}^n$. We say that a cone Γ is *solid* if $\operatorname{int} \Gamma \neq \varnothing$. A closed solid convex cone is said to be *acute* if there exists a hyperplane of dimensionality $n - 1$ which meets the cone at its apex only.

Let Γ be a cone in $\mathbf{R}^n$ with apex at zero, $S = \Gamma \setminus \{0\}$. We fix some vector $e \in S$. Everywhere in this chapter t is a real positive variable.

A function $f(x)$ is said to be *regularly varying* at infinity along Γ if it is defined for all $x \in S$, $|x| \geq c > 0$, is positive and measurable on this set, and

$$f(tx)/f(te) \to \varphi(x) > 0, \qquad \varphi(x) < \infty \tag{1.1.1}$$

for any $x \in S$ $t \to \infty$.

We observe that the set of all functions which are regularly varying at infinity along Γ does not depend on the vector e (let $R_1(\Gamma)$ denote this set). It is easily seen indeed that if (1.1.1) holds for f then for any $b \in S$

$$\frac{f(tx)}{f(tb)} = \frac{f(tx)}{f(te)}\frac{f(te)}{f(tb)} \to \frac{\varphi(x)}{\varphi(b)}. \tag{1.1.2}$$

for any $b \in S$ as $t \to \infty$. In accordance with (1.1.1), we set $\varphi = H_e(f)$. A function $L \in R_1(\Gamma)$ is said to be *slowly varying* at infinity along Γ if $H_e(L) \equiv 1$. In the one-dimensional case, as the cone Γ we consider the set of all non-negative real numbers, and the functions which are regularly (slowly) varying at infinity along this set are merely referred to as regularly (slowly) varying without specifying the set along which they are regularly (slowly) varying. Let $R_2(\Gamma)$ denote the set of all $f \in R_1(\Gamma)$ such that

$$\frac{f(tx_t) - f(tx)}{f(te)} \to 0 \tag{1.1.3}$$

for an arbitrary family of vectors $x_t \in S$, $x_t \to x \in S$ as $t \to \infty$. A function φ on S is said to be *homogeneous* in S with homogeneity degree $\rho \in R$ if it is Borel-measurable on S and $\varphi(tx) = t^\rho \varphi(x)$ for any $x \in S$, $t > 0$. In view of this definition, we set

$\rho = \operatorname{ind}\varphi$ and let $O_1(\Gamma)$ stand for the set of all homogeneous positive functions on S. Let $O_2(\Gamma)$ denote the set of all continuous on S functions $\varphi \in O_1(\Gamma)$. For $i = 1, 2$, we set $T_i(\Gamma) = \{L : L \in R_i(\Gamma), H_e(L) \equiv 1\}$ (in view of (1.1.2), the sets $T_i(\Gamma)$ do not depend on the vector $e \in S$ as well).

THEOREM 1.1.1. *In the case where* $i = 1, 2$, *for any function* φ *on* S *there exists a function* $f \in R_i(\Gamma)$ *such that* $\varphi = H_e(f)$ *if and only if* $\varphi \in O_i(\Gamma)$ *and* $\varphi(e) = 1$.

PROOF. Let $i = 1$. We assume that $\varphi = H_e(f)$ for some function $f \in R_1(\Gamma)$. We set $g(t) = f(te)$; by virtue of (1.1.1), $g(t)$ is regularly varying at infinity, therefore, by virtue of Theorem 1.3 in (Seneta, 1976),

$$g(\lambda t)/g(t) \to \lambda^\rho$$

as $t \to \infty$ for any $\lambda > 0$ and some real ρ. Furthermore, for any $x \in S$ and $\lambda > 0$ by (1.1.1) as $t \to \infty$

$$\frac{f(\lambda tx)}{f(tx)} = \frac{f(\lambda tx)}{f(te)}\frac{f(te)}{f(tx)} \to \frac{\varphi(\lambda x)}{\varphi(x)}, \tag{1.1.4}$$

$$\frac{f(\lambda tx)}{f(tx)} = \frac{f(tx\lambda)}{f(\lambda te)}\frac{f(\lambda te)}{f(te)}\frac{f(te)}{f(tx)} \to \frac{\varphi(x)}{\varphi(x)}\lambda^\rho = \lambda^\rho. \tag{1.1.5}$$

Comparing (1.1.4) and (1.1.5), we see that $\varphi(\lambda x) = \lambda^\rho\varphi(x)$, that is, φ is homogeneous. The measurability, positivity of φ and the equality $\varphi(e) = 1$ immediately follow from (1.1.1). Conversely, let $\varphi \in O_1(\Gamma)$ and $\varphi(e) = 1$. We set $f(x) = \varphi(x)$, $x \in S$. In this case,

$$\frac{f(tx)}{f(te)} = \frac{\varphi(tx)}{\varphi(te)} = \frac{t^\rho\varphi(x)}{t^\rho\varphi(e)} = \varphi(x),$$

where $\rho = \operatorname{ind}\varphi$, which implies that $\varphi = H_e(f)$. Now let $i = 2$. If $\varphi = H_e(f)$, where $f \in R_2(\Gamma)$, then with the use of the just proved part of the theorem for $i = 1$ we find that $\varphi(e) = 1$ and $\varphi \in O_1(\Gamma)$. Let us demonstrate that φ is continuous.

If a sequence $c_k \in S$ is chosen in such a way that $c_k \to c \in S$ as $k \to \infty$, then, by virtue of (1.1.1) and (1.1.3), for any $\varepsilon > 0$ and $k \in \mathbf{N}$ there exists t_k such that

$$\left|\frac{f(c_k t)}{f(te)} - \varphi(c_k)\right| < \varepsilon/2 \tag{1.1.6}$$

for $t \geq t_k$.

Without loss of generality we assume that $t_k \uparrow \infty$ as $k \to \infty$. We set $x_t = c_k$ for $t \in [t_k, t_{k+1})$. We observe that $x_t \to c$ as $t \to \infty$. It is easily seen indeed that if k_0 is chosen so that $|c - c_k| < \delta$ as soon as $k \geq k_0$, then for $t > t_{k_0}$ from the definition of x_t it follows that $|x_t - c| < \delta$. Therefore, by (1.1.1) and (1.1.3) there exists $t_0 > 0$ such that

$$\left|\frac{f(tx_t)}{f(te)} - \varphi(c)\right| < \varepsilon/2 \tag{1.1.7}$$

for $t \geq t_0$.

Setting $t = t_k$ in (1.1.6) and (1.1.7), and making use of the triangle inequality, we obtain

$$|\varphi(c_k) - \varphi(c)| \le \left| \frac{f(c_k t_k)}{f(t_k e)} - \varphi(c_k) \right| + \left| \frac{f(c_k t_k)}{f(t_k e)} - \varphi(e) \right| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for $k \ge \max(k_0, k_1)$, where $k_1 = \min\{k : t_k \ge t_0, k \in \mathbf{N}\}$. Thus, φ is continuous. Conversely, if $\varphi(e) = 1$ and $\varphi \in O_2(\Gamma)$, then

$$\frac{\varphi(tx_t) - \varphi(tx)}{\varphi(te)} = \frac{t^\rho \varphi(x_t) - t^\rho \varphi(x)}{t^\rho \varphi(e)} = \varphi(x_t) - \varphi(x) \to 0$$

for $t > 0$, $x_t \in S$, $x_t \to x \in S$, as $t \to \infty$. where $\rho = \operatorname{ind} \varphi$. Therefore, $\varphi \in R_2(\Gamma)$ and $H_e(\varphi) = \varphi$. The theorem is thus proved. □

We set $B = \{x : x \in S, |x| = 1\}$.

THEOREM 1.1.2. (a) *If* $f \in R_2(\Gamma)$, *then relation* (1.1.1) *holds for* f *uniformly in* $x \in K$ *on an arbitrary compact* $K \subseteq S$.

(b) $R_1(\Gamma) = R_2(\Gamma)$ *if and only if* B *is finite.*

Assertion (a) of this theorem extends the well-known theorem on uniform convergence of one-dimensional regularly varying functions (see, e.g., (Seneta, 1976; Bingham *et al.*, 1987)).

PROOF. Let us prove part (a). Let $L \in T_2(\Gamma)$. We assume that the relation

$$\frac{L(tx)}{L(te)} \to 1, \qquad t \to \infty,$$

holds not uniformly in $x \in K$ on some compact $K \subseteq S$. Then there exist $\varepsilon > 0$ and sequences $c_k \in K$, $t_k \ge 0$ such that

$$\left| \frac{L(c_k t_k)}{L(t_k e)} - 1 \right| > \varepsilon. \tag{1.1.8}$$

Without loss of generality we assume that $c_k \to c \in K$ and $t_k \uparrow \infty$ as $k \to \infty$. We set $x_t = c_k$ for $t \in [t_k, t_{k+1})$, then, because $x_t \to c$ as $t \to \infty$, there exists $t_0 > 0$ such that

$$|L(tx_t)/L(te) - 1| < \varepsilon \tag{1.1.9}$$

for $t \ge t_0$. Let k be chosen in such a way that $t_k > t_0$, then, upon setting $t = t_k$ in (1.1.9), we obtain

$$|L(c_k t_k)/L(t_k e) - 1| < \varepsilon,$$

which contradicts (1.1.8). Let $f \in R_2(\Gamma)$ and $\varphi = H_e(f)$. We set $L(x) = f(x)/\varphi(x)$. We observe that

$$\frac{L(ta_t)}{L(te)} = \frac{\varphi(te)}{\varphi(ta_t)} \frac{f(ta_t)}{f(te)} \to \frac{\varphi(a)}{\varphi(a)} = 1$$

for an arbitrary family of $a_t \in S$, $a_t \to a \in S$ as $t \to \infty$ which immediately implies that $L \in T_2(\Gamma)$. Furthermore, on an arbitrary compact $K \subseteq S$, as $t \to \infty$,

$$|f(xt)/f(te) - \varphi(x)| = \varphi(x)|L(xt)/L(te) - 1| \to 0$$

uniformly in $x \in K$ because φ is bounded on the compact K as a continuous on K function.

Let us turn to the proof of part (b). If B is finite, then f reduces to a finite number of regularly varying functions of a single variable, therefore, in this case $R_1(\Gamma) = R_2(\Gamma)$ (see (Seneta, 1976)). Let B be infinite and a be a limit point of B. For $t > 0$ and $x \in B$ we set

$$L(tx) = l(t)(1 + \min(1, 1/t|x - a|)), \tag{1.1.10}$$

where $l(t)$ is some slowly varying function of one variable. From (1.1.10) it follows that $L \in R_1(\Gamma)$. Nevertheless,

$$\sup_{x \in B} L(tx)/l(t) = 2,$$

so $L \notin R_2(\Gamma)$. The theorem is thus proved. □

LEMMA 1.1.1. *Let*

$$f(x) = \exp\left(\alpha_m + (\alpha_{m+1} - \alpha_m) \int_0^{\{c \ln x\}} g(u)\, du\right), \qquad x \geq 1,$$

where $c > 0$, $m = [c \ln x]$, *let the numerical sequence* $(\alpha_k,\ k \in \mathbf{N})$ *behave so that* $\alpha_{k+1} - \alpha_k \to 0$ *as* $k \to \infty$, *and let* g *be a non-negative compactly supported function whose support lies in the interval* $(0, 1)$ *and* $\int_0^1 g(u)\, du = 1$. *Then* $f(x)$ *is infinitely differentiable for* $x \geq 1$, *and for any* $k \in \mathbf{N}$

$$\frac{df^k(x)}{dx^k} \equiv f^{(k)}(x) = o(x^{-k} f(x)) \tag{1.1.11}$$

as $x \to \infty$.

PROOF. We observe that

$$f(x) = \exp(h(c \ln x)), \qquad x \geq 1, \tag{1.1.12}$$

where

$$h(x) = \alpha_m + (\alpha_{m+1} - \alpha_m) \int_0^{\{x\}} g(u)\, du, \qquad x \geq 0.$$

Let x be not integer, then

$$h'(x) = (\alpha_{m+1} - \alpha_m) g(\{x\}).$$

If x is an integer, then $h'(x) = 0$, therefore, the formula

$$h'(x) = (\alpha_{m+1} - \alpha_m) g(\{x\}), \qquad x \geq 0$$

is true. Subsequent differentiating of the last relation yields for all $k \in \mathbf{N}$

$$h^{(k)}(x) = (\alpha_{m+1} - \alpha_m) g^{(k-1)}(\{x\}),$$

where $g^{(0)}(x) = g(x)$. Therefore, for all $k \in \mathbf{N}$, as $x \to \infty$,

$$h^{(k)}(x) = o(1). \tag{1.1.13}$$

We set

$$v(x) = h(c \ln x), \qquad x \geq 1.$$

By formula (0.430) in (Gradshteyn, Ryzhik, 1980), for all $k \in \mathbf{N}$

$$v^{(k)}(x) = x^{-k} \sum_{i=1}^{k} b_{ik} g^{(i)}(c \ln x), \tag{1.1.14}$$

where b_{ik} are some numerical coefficients which do not depend on x. From (1.1.13) and (1.1.14) it follows that for all $k \in \mathbf{N}$

$$v^{(k)}(x) = o(x^{-k}) \tag{1.1.15}$$

as $x \to \infty$. From (1.1.12) we find that $f(x) = \exp(v(x))$, $x \geq 1$. By formula (0.430) in (Gradshteyn, Ryzhik, 1980), for any $k \in \mathbf{N}$

$$f^{(k)}(x) = f(x) \sum C(i_1, \ldots, i_k) \prod_{j=1}^{k} (v^{(j)}(x))^{i_j}, \tag{1.1.16}$$

where the summation is over all integers $i_1, \ldots, i_k \geq 0$ such that $\sum_{j=1}^{k} j i_j = k$ and $C(i_1, \ldots, i_k)$ are some constants. To complete the proof it suffices to say that (1.1.11) immediately follows from (1.1.15) and (1.1.16). □

LEMMA 1.1.2. *Let a cone* Γ *be closed, then for any function* $L \in T_2(\Gamma)$ *there exists a real* $a > 0$ *such that*

$$0 < \inf_{x \in A} L(x) \leq \sup_{x \in A} L(x) < \infty \tag{1.1.17}$$

for an arbitrary bounded set $A \subseteq \{x : x \in S, |x| \geq a\}$.

PROOF. By virtue of Theorem 1.1.2, for an arbitrary $0 < \varepsilon < 1$ there exists $b > 0$ such that

$$L(|x|e)(1 - \varepsilon) \leq L(x) \leq L(|x|e)(1 + \varepsilon)$$

for $|x| \geq b$. Since the function $g(t) = L(te)$ of one variable is slowly varying at infinity and Lemma 1.1.2 is known to be true for $n = 1$ (Seneta, 1976), from the last inequalities we find that (1.1.17) holds for some $a \geq b$. □

Let Z^n denote the set of all vectors in $\mathbf{R}^n$ with non-negative integer coordinates. For $k = (k_1, \ldots, k_n) \in Z^n$, $k \neq 0$, and an infinitely differentiable function $f(x)$ in $\mathbf{R}^n$ we set

$$f^{(k)}(x) = \frac{\partial^j f(x)}{\partial x_{i_1}^{k_{i_1}} \ldots \partial x_{i_m}^{k_{i_m}}},$$

where $j = k_1 + \cdots + k_n$ and $k_{i_1}, \dots, k_{i_m}$ are the positive components of the vector k, $1 \le i_1 < \cdots < i_m \le n$, and $f^{(0)}(x) = f(x)$. For a real $a > 0$ we set

$$a^k = a^{k_1 + \cdots + k_n}.$$

THEOREM 1.1.3. *Let a cone* Γ *be closed and* $a > 0$ *be chosen so that the function* $L \in T_2(\Gamma)$ *obeys* (1.1.17). *Then for any* α, β, $0 < \alpha < 1 < \beta < \infty$, *there exist two infinitely differentiable functions* $L_i(x) \in T_2(\mathbf{R}^n)$, $i = 1, 2$, *such that*

(a) *for any* $k \in Z^n$, $k \neq 0$, $i = 1, 2$

$$L_i^{(k)}(x) = o(|x|^{-k} L_i(x))$$

for $|x| \to \infty$;

(b) *for any* $x \in S$, $|x| \ge a$

$$\inf_{y \in K_x} L(y) \le L_1(x) \le L(x) \le L_2(x) \le \sup_{y \in K_x} L(y),$$

where $K_x = \{y\colon y \in S, \alpha|x| \le |y| \le \beta|x|, |y| \ge a\}$;

(c) $L_i(x)/L(x) \to 1$ *for* $|x| \to \infty$, $i = 1, 2$;

(d) $L_i(x) = L_i(y)$ *for* $|x| = |y|$, $i = 1, 2$.

The fact that for any slowly varying function $L(t)$ of one variable there exists an infinitely differentiable function $L_1(t)$ which is equivalent to the former at infinity and $L_1^k(t) = o(t^{-k} L_1(t))$ as $t \to \infty$ is well known (Bingham *et al.*, 1987). Nevertheless, this theorem seems to be new even in the one-dimensional case.

PROOF. Without loss of generality we assume that $a = 1$ (if it is not the case, we turn to the function $\widetilde{L}(x) = L(ax)$, $|x| \ge 1$). We set $L_2(x) = f(|x|)$, where

$$f(t) = \exp\left(\alpha_m + (\alpha_{m+1} - \alpha_m) \int_0^{\{c \ln t\}} g(u)\, du\right), \tag{1.1.18}$$

$c > 0$, $t \ge 1$, $m = [c \ln t]$, $g(u)$ is an arbitrary non-negative compactly supported function with support in the interval $(0, 1)$ normalised by the condition

$$\int_0^1 g(u)\, du = 1,$$

and

$$\alpha_m = \sup \ln L(x), \quad x \in S, \quad |x| \ge 1, \quad \exp((m-1)/c) \le |x| \le \exp((m+1)/c).$$

By virtue of Theorem 1.1.2, $\alpha_{k+1} - \alpha_k \to 0$ as $k \to \infty$. Therefore, by Lemma 1.1.1,

$$f^{(j)}(t) = o(t^{-j} f(t)) \tag{1.1.19}$$

for any $j \in \mathbf{N}$ as $t \to \infty$. We observe also that

$$L_2^{(k)}(x) = \sum_{i=1}^{l} |x|^{i-2l} f^{(i)}(|x|) P_{ik}(x) \tag{1.1.20}$$

for any $k \in Z^n$, $k \neq 0$, where $P_{ik}(x)$ are polynomials of degrees at most l, $l = k_1 + \cdots + k_n$. It is easily seen indeed that

$$\frac{\partial L_2(x)}{\partial x_j} = L_2^{(e_j)}(x) = \frac{f^{(1)}(|x|)}{|x|} x_j$$

for any $j = 1, \ldots, n$. Further, if (1.1.20) holds true for some $k \in Z^n$, $k \neq 0$, then upon setting $k' = k + e_j$ we obtain

$$L_2^{(k')}(x) = \sum_{i=1}^{l} \left(\frac{f^{(i+1)}(|x|)}{|x|^{2l-i+1}} x_j P_{ik}(x) + \frac{f^{(i)}(|x|)}{|x|^{2l-i+2}} (i - 2l) x_j P_{ik}(x) \right) + \sum_{i=1}^{l} \frac{f^{(i)}(|x|)}{|x|^{2l-i}} P_{ik}^{(e_j)}(x)$$

Since

$$\frac{P_{ik}^{(e_j)}(x)}{|x|^{2l-i}} = \frac{(x, x) P_{ik}^{(e_j)}(x)}{|x|^{2l+2-i}}$$

and the degree of the polynomial $(x, x) P_{ik}^{(e_j)}(x)$ does not exceed $l+1$, we see that $L_2^{k'}(x)$ is of the desired form, and (1.1.20) is proved. Now validity of assertion (a) of the theorem for $i = 2$ immediately follows from (1.1.19) and (1.1.20). From the explicit formula (1.1.18) we arrive at

$$\min(\exp(\alpha_m), \exp(\alpha_{m+1})) \leq L_2(x) \leq \max(\exp(\alpha_m), \exp(\alpha_{m+1})),$$

which yields

$$L(x) \leq L_2(x) \leq \sup(L(y), \exp((m-1)/c) \leq |y| \leq \exp((m+2)/c), |y| \geq 1),$$

where $m = [c \ln |x|]$. In order to prove the inequalities in part (b) concerning the function L_2 it suffices to say that the inequalities

$$\alpha |x| \leq \exp((m-1)/c) \leq \exp((m+2)/c) \leq \beta |x|$$

hold for $c \geq \max(-1/\ln \alpha, 2/\ln \beta)$, $|x| \geq 1$. Validity of assertion (c) follows from the above bounds and the fact that

$$L(x) = (1 + o(1)) \sup_{y \in K_x} L(y)$$

by virtue of Theorem 1.1.2 as $|x| \to \infty$. Validity of assertion (d) follows from formula (1.1.18) for L_2. All statements of the theorem about L_1 are proved by word-for-word repetition of the reasoning concerning L_2 we have just carried out. The theorem is thus proved. □

In what follows the cone Γ is assumed to be closed. The lemma below is an extension of the corresponding one-dimensional results (Seneta, 1976; Bingham *et al.*, 1987).

LEMMA 1.1.3. *Let* $L \in T_2(\Gamma)$ *and* (1.1.17) *hold for* $|x| \geq a > 0$. *Then for* $\varepsilon > 0$ *the functions*

$$L^*(x) = |x|^{-\varepsilon} \sup_{a \leq |y| \leq |x|} |y|^{\varepsilon} L(y),$$
$$L^{**}(x) = |x|^{\varepsilon} \inf_{a \leq |y| \leq |x|} |y|^{-\varepsilon} L(y)$$

are slowly varying at infinity along Γ, *and*

$$L(x) = (1 + o(1))L^*(x) = (1 + o(1))L^{**}(x)$$

as $|x| \to \infty$, $x \in S$.

PROOF. Let L_1 and L_2 be infinitely differentiable functions that approximate L such that assertions (a)–(d) of Theorem 1.1.3 are true. We set $f_1(t) = L_1(x)$, $f_2(t) = L_2(x)$, where $t = |x|$. Then

$$L^*(x) \leq t^{-\varepsilon} \sup_{a \leq u \leq t} u^{\varepsilon} f_2(u) \leq f_2(t) = L_2(x)$$

for $t \geq t_0$, $t_0 \geq a$, because

$$(t^{\varepsilon} f_2(t))' = t^{\varepsilon - 1} f_2(t) \left(\varepsilon + \frac{f_2'(t)}{t f_2(t)} \right) \geq 0$$

for sufficiently large t. By the same token, $L^*(x) \geq L_1(x)$ for sufficiently large $|x|$. Therefore, $L^*(x) = (1 + o(1))L(x)$ as $|x| \to \infty$. The statement concerning L^{**} is proved in the same way as above. □

The theorem below is known in the one-dimensional case as the integral representation theorem for slowly varying functions (Seneta, 1976; Bingham *et al.*, 1987).

THEOREM 1.1.4. *The inclusion* $L \in T_2(\Gamma)$ *holds if and only if there exist constants* $a > 0$, $c \in R$, *and measurable functions* $\eta(x)$, $\varepsilon(t)$ *defined for* $x \in S$, $|x| \geq a$ *and* $t \geq a$, *respectively, such that*

$$L(x) = \exp \left(\eta(x) + \int_a^{|x|} \frac{\varepsilon(u)}{u} du \right),$$

for $x \in S$, $|x| \geq a$, $\eta(x) \to c$ *as* $|x| \to \infty$, *and*

$$\varepsilon^{(k)}(t) = o(t^{-k})$$

for any integer $k \geq 0$ *as* $t \to \infty$.

PROOF. Let $L \in T_2(\Gamma)$. We represent L as

$$L(x) = \exp(\eta(x) + \ln f(|x|)),$$

where

$$\eta(x) = \ln\left(\frac{L(x)}{L_2(x)}\right),$$

assertions (a)–(d) of Theorem 1.1.3 are true for L_2, and $f(|x|) = L_2(x)$. By virtue of assertion (c) of Theorem 1.1.3, $\eta(x) \to 0$ as $|x| \to \infty$. By virtue of (1.1.15), $v^{(k)}(t) = o(t^{-k})$ for any $k \in \mathbf{N}$ as $t \to \infty$, where $v(t) = \ln f(t)$. We set $\varepsilon(t) = tv'(t)$; then

$$L(x) = \exp\left(\eta(x) + v(a) + \int_a^{|x|} \frac{\varepsilon(u)}{u}\, du\right),$$

for $|x| \geq a$, where the constant a is defined in Theorem 1.1.3, and $\varepsilon^{(k)}(t) = o(t^{-k})$ for any integer $k \geq 0$ as $t \to \infty$. Now let us prove the sufficiency: for $x, e \in S$

$$\frac{L(tx)}{L(te)} = \exp\left(\eta(tx) - \eta(te) + \int_{t|e|}^{t|x|} \frac{\varepsilon(u)}{u}\, du\right),$$

which yields

$$\frac{L(tx)}{L(te)} = (1 + o(1)) \exp\left(\int_{|e|}^{|x|} \frac{\varepsilon(ty)}{y} dy\right) = 1 + o(1)$$

as $t \to \infty$. The theorem is thus proved. □

1.2. Weak convergence of measures and functions

In this section we assume that Γ is a closed convex acute solid cone with apex at zero. We set $G = \operatorname{int}\Gamma$. We introduce an *order relation* on Γ (Vladimirov, 1978): we write $x \overset{\Gamma}{\leq} y$ and $x \overset{\Gamma}{<} y$, if $x, y, y - x \in \Gamma$ and $x \in \Gamma, y, y - x \in G$ respectively. We omit the specification of the cone and write $x \leq y$, $x < y$ where this does not lead to ambiguity. A real-valued function $f(x)$ defined on an open set $\sigma \subseteq \Gamma$ is said to be a *monotone non-decreasing* (*non-increasing*, respectively) in Γ if $f(x) \leq f(y)$ ($f(x) \geq f(y)$, respectively) for $x < y$, $x, y \in \sigma$. The notation $\lim_{y\downarrow x} f(y) = b$ ($\lim_{y\uparrow x} f(y) = b$) means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(y) - b| < \varepsilon$ for any $y \in \sigma$, $|y - x| < \delta$, $x < y$ ($y < x$, respectively).

Let $M_1(\sigma)$, $M_2(\sigma)$ denote, respectively, the sets of all non-decreasing and of all non-increasing in Γ functions defined on a set $\sigma \subseteq \Gamma$ (we do not specify the dependence of the sets $M_1(\sigma)$ and $M_2(\sigma)$ on Γ because the cone Γ is assumed to be fixed throughout this chapter), $M(\sigma) = M_1(\sigma) \cup M_2(\sigma)$.

LEMMA 1.2.1. *For $x, y \in \Gamma$, $x < y$ the sets*

$$A = \{z : z \in \Gamma,\ x < z < y\}, \qquad B = \{z : z \in \Gamma,\ x < z\}$$

are non-empty open subsets of Γ.

PROOF. The sets A and B are not empty because $\frac{x+y}{2} \in A \cap B$. The set B is open because $B = \{z : z = x + u, u \in G\}$. Let $z \in A$. Since G is open, there exists $\delta_1 > 0$ such that $y + z - h \in G$ for all $h \in \mathbf{R}^n$, $|h| < \delta_1$. Since B is open, there exists $\delta_2 < \delta_1$ such that $x < u$ for all $u \in \mathbf{R}^n$, $|u - z| < \delta_2$. Then $x < u$ and $y - u = y - z + z - u \in G$ for all $u \in \mathbf{R}^n$, $|u - z| < \delta_2$; hence it follows that $u \in A$ for $|u - z| < \delta_2$. The lemma is proved. □

LEMMA 1.2.2. *Let $f \in M_1(\sigma)$ ($f \in M_2(\sigma)$).*

(a) *Then for any $x \in \sigma$ there exist two limits*

$$\lim_{y \uparrow x} f(y) = f(x_-), \tag{1.2.1}$$

$$\lim_{y \downarrow x} f(y) = f(x_+), \tag{1.2.2}$$

where

$$f(x_-) = \sup(f(y), y \in \sigma, y < x), \tag{1.2.3}$$

$$f(x_+) = \inf(f(y), y \in \sigma, x < y) \tag{1.2.4}$$

($f(x_-) = \inf(f(y), y < x, y \in \sigma)$, $f(x_+) = \sup(f(y), x < y, y \in \sigma)$, respectively), and

$$f(x_-) \le f(x) \le f(x_+) \tag{1.2.5}$$

($f(x_-) \ge f(x) \ge f(x_+)$, respectively).

(b) *for f to be continuous at $x \in \sigma$, it is necessary and sufficient that $f(x_+) = f(x_-)$.*

PROOF. Let f be monotone non-decreasing in Γ. Then by (1.2.3) and (1.2.4) for any $\varepsilon > 0$ there exist $x_\varepsilon, y_\varepsilon \in \sigma$, $x_\varepsilon < x < y_\varepsilon$ such that

$$\varepsilon \ge f(x_-) - f(x_\varepsilon) \ge 0, \tag{1.2.6}$$

$$\varepsilon \ge f(y_\varepsilon) - f(x_+) \ge 0. \tag{1.2.7}$$

By virtue of Lemma 1.2.1, the set $U_\varepsilon = \{y : x_\varepsilon < y < y_\varepsilon\}$ is open and non-empty, and therefore, so is the set $V_\varepsilon = U_\varepsilon \cap \sigma$, because σ is open and $x \in V_\varepsilon$. Therefore, $f(x_\varepsilon) \le f(y) \le f(x_-)$ for any $y \in V_\varepsilon$, $y < x$, from the monotonicity of f and (1.2.3). Taking into account (1.2.6), we obtain

$$|f(y) - f(x_-)| \le \varepsilon, \quad y \in V_\varepsilon, \quad y < x. \tag{1.2.8}$$

For $y \in V_\varepsilon$ and $y > x$, it follows that $f(x_+) \le f(y) \le f(y_\varepsilon)$ from the monotonicity of f and (1.2.4). Taking into account (1.2.7), we obtain

$$|f(y) - f(x_+)| \le \varepsilon, \quad y \in V_\varepsilon, \quad y > x. \tag{1.2.9}$$

Since ε is chosen arbitrarily, (1.2.1) and (1.2.2) immediately follow from (1.2.8) and (1.2.9) respectively. We observe that $f(z) \le f(x) \le f(y)$ for $z, y \in \sigma$, $z < x < y$. So (1.2.5) follows from (1.2.3), (1.2.4), and the last relation.

Now let, in addition, $f(x_+) = f(x_-)$. Then, because f is monotone, $f(x_\varepsilon) \le f(y) \le f(y_\varepsilon)$ for $y \in V_\varepsilon$. Taking into account relations (1.2.6) and (1.2.7), we find that $|f(x) - f(y)| \le \varepsilon$ for $y \in V_\varepsilon$, because from inequalities (1.2.5) and $f(x_-) = f(x_+)$ it immediately follows that $f(x_-) = f(x) = f(x_+)$. Conversely, let $f(y)$ be continuous at a point $x \in \sigma$. Then from relations (1.2.1) and (1.2.2) just proved we find that $f(x_-) = f(x) = f(x_+)$.

If f is monotone non-increasing, then $-f$ is monotone non-decreasing, and all statements of Lemma 1.2.2 concerning f immediately follow from the fact that they are true for $-f$. The lemma is thus proved. □

For $i = 1, 2$ we set

$$M_i^+(\sigma) = \{f \colon f \in M_i(\sigma), f(x_+) = f(x)\ \forall x \in \sigma\},$$
$$M^+(\sigma) = M_1(\sigma) \cup M_2(\sigma).$$

We say that a sequence $f_k \in M_i^+(\sigma)$ *weakly converges* as $k \to \infty$ to $f \in M_i^+(\sigma)$ and write

$$f_k \Rightarrow f, \qquad k \to \infty,$$

if $f_k(x) \to f(x)$ as $k \to \infty$ at an arbitrary continuity point x of the function f.

LEMMA 1.2.3. *If $f_k \Rightarrow f$ and $f_k \Rightarrow g$ as $k \to \infty$, then $f(x) = g(x)$ for $x \in \sigma$.*

PROOF. We consider the discontinuity points of the function f in the set $l_x = \{y \colon y \in \sigma, y = tx, t > 0\}$ for some $x \in \sigma$. There are at most countably many of them, because, as we easily see, the function $\varphi(t) = f(tx)$ is a monotone function of real variable t and hence has at most countably many discontinuity points. If φ is continuous at the point t, then by virtue of Lemma 1.2.2

$$f(tx_-) = \lim_{s \uparrow t} \varphi(s) = \varphi(t),$$
$$f(tx_+) = \lim_{s \downarrow t} \varphi(s) = \varphi(t),$$

which implies that $f(tx_-) = f(tx_+)$, and using Lemma 1.2.2 again we find that f is continuous at the point tx. From the definition of weak convergence it follows that $f(y) = g(y)$ at those points y where f and g are continuous, hence f and g coincide everywhere on the set l_x except for, maybe, a countable number of points. We choose a sequence $t_k \downarrow 1$ as $k \to \infty$ so that

$$f(t_k x) = g(t_k x). \tag{1.2.10}$$

If we let k grow without bound in (1.2.10), with the use of Lemma 1.2.2 we obtain

$$f(x_+) = g(x_+),$$

hence $f(x) = g(x)$ because $f, g \in M_i^+(\sigma)$. The lemma is thus proved. □

For $i = 1, 2$ a family of functions $F \subseteq M_i^+(\sigma)$ is said to be *weakly relatively compact* or *weakly pre-compact* if one can extract a weakly converging sequence from any sequence in F.

We prove the following analogue of the Helly theorem.

THEOREM 1.2.1. *Let either $i = 1$ or $i = 2$. For a family of functions $F \subseteq M_i^+(G)$ to be weakly relatively compact it is necessary and sufficient that*

$$\sup_{f \in F} |f(x)| < \infty \tag{1.2.11}$$

for any $x \in G$.

PROOF. We observe that the weak pre-compactness of a family $F \subseteq M_2^+(G)$ is equivalent to the weak pre-compactness of the family $F_1 = \{g : g = -f,\ f \in F\} \subseteq M_1^+(G)$, and (1.2.11) holds true if and only if $\sup_{g \in F_1} |g(x)| < \infty$, so without loss of generality we assume that $i = 1$. For fixed $x, y \in G$, $x < y$, we set $\sigma = \{z : x < z < y\}$ so F_2 is the set of all restrictions of the functions f in F to the set σ. In this case, $F_2 \subseteq M_1^+(\sigma)$, because $F \subseteq M_1^+(G)$. Let us first demonstrate that the weak pre-compactness of F_2 follows from (1.2.11). Let Q be the set of all vectors in σ with rational coordinates, $Q = \{r_1, r_2, \dots\}$, $a_k = (a_k^1, a_k^2, \dots) = (f_k(r_1), f_k(r_2), \dots)$, where $k \in \mathbf{N}$ and $(f_k,\ k \in \mathbf{N})$ is a fixed sequence in F_2. We note that $f(x) \le f(z) \le f(y)$ for any $f \in F_2$ and $z \in \sigma$. Therefore, in view of (1.2.11) there exists a constant c which does not depend on $f \in F_2$ and $z \in \sigma$ such that $|f(z)| < c$ for $f \in F_2$ and $z \in \sigma$. From the last note it immediately follows that $|a_k^j| \le c$ for any $j, k \in \mathbf{N}$. By the known criterion of pre-compactness in $\mathbf{R}^\infty$ (Billingsley, 1968), there exists a sequence of positive integers $(k(j),\ j \in \mathbf{N})$ such that

$$a_{k(j)} \to a \in \mathbf{R}^\infty$$

as $j \to \infty$. What this means in terms of the functions f_k is that

$$f_{k(j)}(r) \to \varphi(r) \tag{1.2.12}$$

for any $r \in Q$ and some function φ on Q as $j \to \infty$. We set

$$f(x) = \inf(\varphi(r) : x < r,\ r \in Q). \tag{1.2.13}$$

We observe that $f \in M_1^+(\sigma)$. It is easy to see, indeed, that if $r, s \in Q$ and $r < s$, then $\varphi(r) \le \varphi(s)$ by virtue of (1.2.12). Let $u, v \in \sigma, u < v$, then we choose arbitrary $r, s \in Q$ such that $u < r < v < s$. Since $\varphi(r) \le \varphi(s)$, taking the infimum over $s > v$ in the last inequality we obtain with the use of (1.2.3)

$$\varphi(r) \le f(v). \tag{1.2.14}$$

In order to prove the monotonicity of f it remains to show that

$$f(u) = \inf(\varphi(r),\ u < r < v,\ r \in Q), \tag{1.2.15}$$

because (1.2.14) has been proved for arbitrary $r \in Q$ such that $u < r < v$. Let us prove (1.2.15). We assume that (1.2.15) is not true; then, in view of (1.2.13),

$$\inf(\varphi(r) : u < r,\ r \in Q) < \inf(\varphi(r),\ u < r < v,\ r \in Q).$$

If this is the case, then there exists $r > u$, $r \in Q$, but $r \notin A = \{z : u < z < v,\ z \in Q\}$, such that $\varphi(r) < \varphi(s)$ for any $s \in A$. But there exists $s \in A$ such that $s < r$: the sets

$\{z: u < z < v\}$ and $\{z: u < z < r\}$ are non-empty open ones (Lemma 1.2.1) and the vector $u(1+\varepsilon)$ lies in their intersection for sufficiently small $\varepsilon > 0$ (because $u(1+\varepsilon) > u$ for $\varepsilon > 0$ and $v - u(1+\varepsilon) = (v-u) - \varepsilon u \in G$, $r - u(1+\varepsilon) = (r-u) - u\varepsilon \in G$ for sufficiently small ε), therefore, $\varphi(s) \le \varphi(r)$, which is impossible.

Thus, f is monotone. Next, let us demonstrate that $f(u) = f(u_+)$. By virtue of (1.2.13), for any $\varepsilon > 0$ there exists $r \in Q$, $r > u$, such that $\varphi(r) - \varepsilon < f(u) \le f(_+)$. Therefore, for $x < y < r$, $y \in \sigma$ the relation $f(u_+) - \varepsilon < f(u) \le f(u_+)$ certainly holds. Since ε is arbitrary, we conclude that $f(u) = f(u_+)$. Thus, $f \in M_1^+(\sigma)$. Let us demonstrate that

$$f_{k(j)} \Rightarrow f, \qquad j \to \infty.$$

Let f be continuous at the point $u \in \sigma$, then

$$f_{k(j)}(r) \le f_{k(j)}(u) \le f_{k(j)}(s)$$

for $x < v < r < u < s < y$ and $r, s \in Q$. If we let j grow without bound and make use of (1.2.12), we obtain

$$\limsup_{j\to\infty} f_{k(j)}(u) \le \varphi(s), \tag{1.2.16}$$

$$\liminf_{j\to\infty} f_{k(j)}(u) \ge \varphi(r). \tag{1.2.17}$$

From (1.2.16) it follows that

$$\limsup_{j\to\infty} f_{k(j)}(u) \le \inf(\varphi(s),\ s > x,\ s \in Q) = f(x). \tag{1.2.18}$$

Taking the infimum of the right-hand side of (1.2.17) over all $r \in Q$ and making use of relation (1.2.15), we obtain

$$\liminf_{j\to\infty} f_{k(j)}(u) \ge f(v),$$

hence, in view of continuity of f in u we find that

$$\liminf_{j\to\infty} f_{k(j)}(u) \ge f(u). \tag{1.2.19}$$

From (1.2.18) and (1.2.19) it follows now that

$$f_{k(j)}(u) \to f(u)$$

as $j \to \infty$. Thus, the weak pre-compactness of F_2 is proved.

Let us fix $u, v \in G$, $u < v$. We observe that

$$G = \bigcup_{m\in\mathbf{N}} \sigma_m,$$

where $\sigma_m = \{z: u/m < z < mv\}$. It is easy to see, indeed, that if $z \in G$, then, because G is open, there exists $\delta > 0$ such that $z + h \in G$ and $v + h \in G$ for $|h| < \delta$. But since G is a cone, we see that $mv - z \in G$ if and only if $v - z/m \in G$. Therefore, $z \in \sigma_m$ as soon as $z - u/m \in G$ and $v - z/m \in G$, which is the case if either $\max\left(\frac{|u|}{m}, \frac{|z|}{m}\right) \le \delta$ or $m \ge \delta^{-1}\max(|u|, |z|)$.

Let f_k is an arbitrary sequence in F. We extract from it a subsequence $(f_{k_1(j)}, j \in \mathbf{N})$ such that

$$f_{k_1(j)}(x) \to f_{(1)}(x), \qquad j \to \infty,$$

holds for some function $f_{(1)} \in M_1^+(\sigma_1)$ at any point $x \in \sigma_1$ of continuity of the function $f_{(1)}$. From the sequence $(k_1(j),\ j \in \mathbf{N})$ we extract a subsequence $(k_2(j),\ j \in \mathbf{N})$ such that

$$f_{k_2(j)}(x) \to f_{(2)}(x), \qquad j \to \infty,$$

holds for some function $f_{(2)}(x) \in M_1^+(\sigma_2)$ at any point x of continuity of the function $f_{(2)}$. Continuing this process, we see that

$$f_{k_m(j)}(x) \to f_{(m)}(x), \qquad j \to \infty, \tag{1.2.20}$$

for some subsequence $(k_m(j),\ j \in \mathbf{N})$ at all points x of continuity of some function $f_{(m)} \in M_1^+(\sigma_m)$. By virtue of Lemma 1.2.3, $f_{(m)}(x) = f_{(l)}(x)$ for $m > l$ and $x \in \sigma_l$. We define the function $f(x)$ as follows: $f(x) = f_{(m)}(x)$ for $x \in \sigma_m$. In view of the abovesaid, f is well defined. Let x be a continuity point of f. Then there exists m such that $x \in \sigma_m$, and from (1.2.20) it follows that

$$f_{k_j(j)}(x) \to f_{(m)}(x) = f(x), \qquad j \to \infty,$$

because $(k_j(j),\ j \in \mathbf{N})$ is, by construction, a subsequence of the sequence $(k_m(j),\ j \in \mathbf{N})$, while σ_m is a neighbourhood of the point x where $f_{(m)}(y) = f(y)$, $y \in \sigma_m$, and therefore, $f_{(m)}(y)$ is continuous at the point x as well. Thus,

$$f_{k_j(j)}(x) \Rightarrow f(x), \qquad j \to \infty.$$

Conversely, let F be weakly pre-compact. We regard (1.2.11) as not true. Then there exists $x \in G$ such that

$$|f_k(x)| \to \infty, \qquad k \to \infty,$$

for some sequence $(f_k,\ k \in \mathbf{N}) \subseteq F$. Without loss of generality we assume that

$$f_k \Rightarrow f, \qquad k \to \infty.$$

As we have seen in the proof of Lemma 1.2.3, a monotone function cannot have more than a countable number of discontinuity points along the ray $l_x = \{y \colon y = tx, t > 0\}$, therefore, there are $t_1, t_2, 0 < t_1 < 1 < t_2 < \infty$ such that f is continuous at the points $t_1 x$ and $t_2 x$, and, because $f_k(t_1 x) \le f_k(x) \le f_k(t_2 x)$ for $k \in \mathbf{N}$, if we let k grow without bound, we conclude that

$$f(t_1 x) \le \liminf_{k\to\infty} f_k(x) \le \limsup_{k\to\infty} f_k(x) \le f(t_2 x).$$

This contradiction completes the proof of the theorem. □

All *measures* we meet below are, unless otherwise stated, non-negative, σ-additive, and σ-finite, defined on the δ-ring $\mathfrak{A}$ of all bounded Borel sets in $\mathbf{R}^n$.

We say that a sequence of measures $(\mu_k,\ k \in \mathbf{N})$ *weakly converges* to a measure μ and write $\mu_k \Rightarrow \mu$, $k \to \infty$, if $\mu_k(A) \to \mu(A)$, as $k \to \infty$, for any $A \in \mathfrak{A}$ such that $\mu(\partial A) = 0$ (here and below ∂A stands for the boundary of the set A). We also say that the family of measures M is *weakly relatively compact* or *weakly pre-compact* if it is possible to extract a converging sequence from any sequence of measures in M.

As concerns the weak convergence of measures, statements similar to those proved above for monotone functions are true. Here we give only one (see (Feller, 1966, Chapter VIII.6)).

THEOREM 1.2.2. *A family of measures M is weakly pre-compact if and only if*

$$\sup_{\mu\in M} \mu(A) < \infty$$

for any $A \in \mathfrak{A}$.

The expression 'measure U on Γ' will mean that the measure U is concentrated on Γ.

For a set $A \subseteq \mathbf{R}^n$ and $t > 0$ we set $tA = \{y\colon y = tx,\ x \in A\}$.

Let a measure U be given. We introduce the family of measures $(\Phi_t,\ t > 0)$:

$$\Phi_t(A) = \frac{U(tA)}{\rho(t)}, \tag{1.2.21}$$

where $\rho(t)$ is a regularly varying function of one variable, $\operatorname{ind}\rho = \gamma \geq 0$, $A \in \Lambda$. We set

$$A_t = \{x\colon x \in \mathbf{R}^n,\ |x| < t\}, \quad B_t = \{x\colon X \in \mathbf{R}^n,\ |x| \leq t\}, \quad t > 0.$$

LEMMA 1.2.4. *For some measure Φ, let*

$$\Phi_t \Rightarrow \Phi, \qquad t \to \infty. \tag{1.2.22}$$

Then

$$\Phi(\partial A_u) = \Phi(\partial B_u) = 0$$

for any $u > 0$.

PROOF. For any $u > 0$

$$\liminf_{t\to\infty} \Phi_t(A_u) \leq \Phi(A_u) \leq \Phi(B_u) \leq \limsup_{t\to\infty} \Phi_t(B_u). \tag{1.2.23}$$

Since the boundaries of the sets A_t and A_u do not overlap for $u \neq t$, there always exists $v > 0$ such that $\Phi(\partial A_v) = 0$, and hence, $\Phi(\partial B_v) = 0$. Then by the definition of weak convergence of measures

$$\lim_{t\to\infty} \Phi_t(A_v) = \Phi(A_v) = \Phi(B_v) = \lim_{t\to\infty} \Phi_t(B_v). \tag{1.2.24}$$

By virtue of (1.2.21), for any $u > 0$

$$\liminf_{t\to\infty} \Phi_t(A_u) = \liminf_{t\to\infty} \Phi_t\left(\frac{u}{v}A_v\right) = \liminf_{t\to\infty} \frac{\rho\left(t\frac{u}{v}\right)}{\rho(t)} \Phi_{t\frac{u}{v}}(A_v)$$

$$= \left(\frac{u}{v}\right)^\gamma \lim_{t\to\infty} \Phi_t(A_v), \tag{1.2.25}$$

$$\limsup_{t\to\infty} \Phi_t(B_u) = \left(\frac{u}{v}\right)^\gamma \lim_{t\to\infty} \Phi_t(B_v). \tag{1.2.26}$$

From relations (1.2.23)–(1.2.26) it follows that $\Phi(A_u) = \Phi(B_u)$, that is, $\Phi(\partial A_u) = 0$. The lemma is proved. □

Instead of (1.2.21), (1.2.22) we will sometimes briefly write

$$U(t\cdot)/\rho(t) \Rightarrow \Phi, \qquad t \to \infty.$$

1.3. Multidimensional Tauberian theorems of Karamata type

Let Γ be a closed convex acute solid cone in $\mathbf{R}^n$ with apex at zero (see the opening of Section 1.1). We set

$$T = \Gamma^* \equiv \{y: y \in \mathbf{R}^n, (y, x) \geq 0 \quad \forall x \in \Gamma\}.$$

The cone T is said to be the cone *dual* to the cone Γ. The cone T is a closed acute convex solid one as well (see (Vladimirov, 1978; Vladimirov *et al.*, 1988)). Therefore, $C \equiv \operatorname{int} T \neq \varnothing$. The *Laplace transform* and the *Laplace–Stieltjes transform* of a function u and of a measure U on Γ are labelled, respectively, $\widehat{u}(\lambda)$ and $\widetilde{U}(\lambda)$:

$$\widehat{u}(\lambda) = \int_\Gamma e^{-(\lambda,x)} u(x)\,dx, \qquad \widetilde{U}(\lambda) = \int_\Gamma e^{-(\lambda,x)} U(dx)$$

provided that they exist for $\lambda \overset{T}{>} a$ with some $a \in T$ (we recall that the last inequality means that $\lambda - a \in C$, see the beginning of Section 1.2). For the sake of convenience. all measures are assumed to be concentrated in Γ.

We also set $G = \operatorname{int}\Gamma$, and let $\overline{A}$ denote the closure of a set $A \subseteq \mathbf{R}^n$. The notation $A \ll \Gamma$ means that $\overline{A} \subseteq G = \operatorname{int}\Gamma$. Let $|A|$ stand for the Lebesgue measure of a set $A \subseteq \mathbf{R}^n$. For two functions $f(x)$ and $g(x)$ defined for $x \in \Gamma$, $|x| \geq a$, for some $a \geq 0$, the notation $f(x) \sim g(x)$ as $|x| \to \infty$ means that

$$\frac{f(x)}{g(x)} \to 1, \qquad |x| \to \infty.$$

Let a measure U and a function $u(x) \geq 0$ locally integrable in $\mathbf{R}^n$ with respect to the measure U be given; then $u(x)U(dx)$ denotes the measure V defined by the equality

$$V(A) = \int_A u(x)U(dx)$$

for every bounded measurable set $A \subseteq \mathbf{R}^n$.

The theorem below extends the well-known uniqueness theorem for one-dimensional Laplace transforms of measures (Feller, 1966, Chapter XIII, Section 1, Theorem 1a).

THEOREM 1.3.1. *For measures U and V, let there exist their Laplace transforms $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ for $\lambda \overset{T}{>} a$ with some $a \in T$, and*

$$\widetilde{U}(\lambda) = \widetilde{V}(\lambda) \qquad \forall \lambda \overset{T}{>} a.$$

Then $U = V$.

Theorem 1.3.1 follows from the inversion formula for Laplace transforms (Vladimirov *et al.*, 1988, Chapter I, Section 2.5, formula (5.3)). We give an independent proof.

PROOF. Let

$$H = \mathbf{R}^n + i(a + C) \equiv \{z : z = x + iy,\ x \in \mathbf{R}^n,\ y \overset{T}{>} a\},$$
$$\bar{U}(z) = \int_\Gamma e^{i(z,x)} U(dx), \quad \bar{V}(z) = \int_\Gamma e^{i(z,x)} V(dx), \quad z \in H.$$

The functions $\bar{U}(z)$ and $\bar{V}(z)$ are analytic functions in the domain H (see (Vladimirov, 1964)). By the hypotheses of the theorem,

$$\bar{U}(\lambda) = \bar{V}(\lambda) \equiv \omega(\lambda) \qquad \forall \lambda \overset{T}{>} a.$$

With the use of the uniqueness theorem for analytic functions of many complex variables (Shabat, 1992, p. 32), we see that $\bar{U}(z) = \bar{V}(z)$. Therefore,

$$\int_\Gamma e^{(i\nu,x)} e^{-(\lambda,x)} U(dx) = \int_\Gamma e^{(i\nu,x)} e^{-(\lambda,x)} V(dx) \quad \forall \nu \in \mathbf{R}^n, \quad \lambda \overset{T}{>} a,$$

hence it follows that

$$\int_\Gamma e^{(i\nu,x)} U_1(dx) = \int_\Gamma e^{(i\nu,x)} V_1(dx), \tag{1.3.1}$$

where

$$U_1(dx) = \frac{e^{-(\lambda,x)}}{\omega(\lambda)} U(dx), \qquad V_1(dx) = \frac{e^{-(\lambda,x)}}{\omega(\lambda)} V(dx).$$

But U_1 and V_1 are probabilistic measures, and (1.3.1) implies that their characteristic functions coincide, hence $U_1 = V_1$. Thus,

$$\int_A e^{-(\lambda,x)} U(dx) = \int_A e^{-(\lambda,x)} V(dx) \tag{1.3.2}$$

for any bounded Borel set A. Furthermore, the measure U is absolutely continuous with respect to the measure V. It is easy to see, indeed, that if $B \in \Lambda$ is chosen in such a way that $V(B) = 0$, then by (1.3.2)

$$\int_B e^{-(\lambda,x)} U(dx) = 0,$$

and therefore, $U(B) = 0$, because $e^{-(\lambda,x)} > 0$ for any $x \in B$. Consequently, we obtain $U(dx) = f(x)V(dx)$ for some function $f(x)$. Then from (1.3.2) it follows that

$$\int_A e^{-(\lambda,x)} f(x)V(dx) = \int_A e^{-(\lambda,x)} V(dx),$$

or $e^{-(\lambda,x)} f(x) = e^{-(\lambda,x)}$ V-almost everywhere due to uniqueness up to a set of V-measure zero of the Radon–Nykodym derivative of the measure $e^{-(\lambda,x)}V(dx)$. Hence it follows that $f(x) = 1$ V-almost everywhere, and therefore, $U = V$. The theorem is proved. □

By the restriction of a measure U defined on $\mathfrak{A}$ to a Borel set $\sigma \subseteq \mathbf{R}^n$ is meant the measure V defined by the equality $V(D) = U(D)$ on the δ-ring of all sets $D \in \mathfrak{A}$ such that $D \subseteq \sigma$.

The theorem below extends the well-known continuity theorem for Laplace transforms (Feller, 1966, Chapter XIII, Section 1, Theorem 2a) to the multidimensional case.

THEOREM 1.3.2. *Let* $(U_k,\ k \in \mathbf{N})$ *be a sequence of measures on* Γ.

(1) *If for some* $a \in T$ *there exist the Laplace transforms* $\widetilde{U}_k(\lambda)$ $\forall\lambda \overset{T}{>} a$, $k \in \mathbf{N}$, *and*

$$\widetilde{U}_k(\lambda) \to \omega(\lambda) < \infty \qquad \forall\lambda \overset{T}{>} a, \tag{1.3.3}$$

as $k \to \infty$, *then* $\omega(\lambda)$ *is the Laplace transform of some measure* U *on* Γ, *and* $U_k \Rightarrow U$ *as* $k \to \infty$.

(2) *Conversely, if* $U_k \Rightarrow U$ *as* $k \to \infty$ *and* $\widetilde{U}_k(a)$ *is bounded, then relation* (1.3.3) *is true, and for the measure* U *there exists the Laplace transform* $\widetilde{U}(\lambda) = \omega(\lambda)$ *for* $\lambda \overset{T}{>} a$.

PROOF. We begin with proving part 2. We take a sequence $t_k \uparrow \infty$ such that $U(\partial A_k) = 0$, where $A_k = \{x\colon x \in \mathbf{R}^n,\ |x| \le t_k\}$. Let $U_k^{(m)}$ and $U^{(m)}$ be the restrictions of the measures U_k and U, respectively, to A_m. By the definition of weak convergence of measures, $U_k^{(m)} \Rightarrow U^{(m)}$ for any $m \in \mathbf{N}$ as $k \to \infty$, therefore (Bender, 1974), as $k \to \infty$

$$\int_{A_m} e^{-(\lambda,x)} U_k(dx) \to \int_{A_m} e^{-(\lambda,x)} U(dx), \quad \forall m \in \mathbf{N}, \quad \lambda \overset{T}{>} a. \tag{1.3.4}$$

Therefore, for $\lambda \overset{T}{>} a$

$$\begin{aligned}\int_\Gamma e^{-(\lambda,x)} U_k(dx) &= \lim_{m\to\infty} \int_{A_m} e^{-(\lambda,x)} U(dx) \\ &= \lim_{m\to\infty} \lim_{k\to\infty} \int_{A_m} e^{-(\lambda,x)} U_k(dx) \le \limsup_{k\to\infty} \widetilde{U}_k(a) < \infty. \end{aligned} \tag{1.3.5}$$

For any $k, m \in \mathbf{N}$, $\lambda \overset{T}{>} a$

$$\begin{aligned}|\widetilde{U}_k(\lambda) - \widetilde{U}(\lambda)| \le{}& \int_{B_m} e^{-(\lambda,x)} U_k(dx) + \int_{B_m} e^{-(\lambda,x)} U(dx) \\ &+ \left| \int_{A_m} e^{-(\lambda,x)} U_k(dx) - \int_{A_m} e^{-(\lambda,x)} U(dx) \right|, \end{aligned} \tag{1.3.6}$$

where $B_m = \Gamma \setminus A_m$. For $h = \lambda - a$, as we easily see,

$$\int_{B_m} e^{-(\lambda,x)} U_k(dx) \le \sup_{x \in B_m} e^{-(h,x)} \widetilde{U}_k(a) \le c e^{-\inf((h,x), x \in B_m)} \to 0 \tag{1.3.7}$$

as $m \to \infty$ uniformly in $k \in \mathbf{N}$ for some constant $c < \infty$, because

$$\inf_{x \in B_m} (h, x) \ge t_m \inf_{x \in B} (h, x) \to \infty, \qquad m \to \infty,$$

where $B = \{x : x \in \Gamma, |x| = 1\}$, due to the fact that $h \in C$ and $t_m \uparrow \infty$.

Let $\varepsilon > 0$. We choose $m \in \mathbf{N}$ such that for any $k \in \mathbf{N}$

$$\int_{B_m} e^{-(\lambda,x)} U_k(dx) < \varepsilon/3, \qquad \int_{B_m} e^{-(\lambda,x)} U(dx) < \varepsilon/3$$

(which is possible from relations (1.3.5) and (1.3.7)). For the chosen m, in view of (1.3.4), one can choose k_0 so that for any $k \ge k_0$

$$\left| \int_{A_m} e^{-(\lambda,x)} U_k(dx) - \int_{A_m} e^{-(\lambda,x)} U(dx) \right| < \varepsilon/3.$$

From the last three inequalities and relation (1.3.6) it now follows that $|\widetilde{U}_k(\lambda) - \widetilde{U}(\lambda)| < \varepsilon$ for any $k \ge k_0$, which proves part 2 of the theorem.

Under the hypotheses of part 1 of the theorem,

$$\sup_{k \in \mathbf{N}} \widetilde{U}_k(\lambda) \ge \int_A e^{-(\lambda,x)} U_k(dx) \ge U_k(A) \inf_{x \in A} e^{-(\lambda,x)}$$

for any bounded Borel set $A \subseteq \mathbf{R}^n$ and any $\lambda \overset{T}{>} a$; by virtue of Theorem 1.2.2, hence the weak pre-compactness of the sequence of measures U_k follows. If the subsequence $U_{k(j)} \Rightarrow U$ as $j \to \infty$ for some measure U, then, by the just proved part 2 of the theorem, as $j \to \infty$

$$\widetilde{U}_{k(j)}(\lambda) \to \widetilde{U}(\lambda) \qquad \forall \lambda \overset{T}{>} a,$$

therefore, $\widetilde{U}(\lambda) = \omega(\lambda)$, and by Theorem 1.3.1 the sequence U_k has a unique limit point U in the sense of weak convergence, and $\widetilde{U}(\lambda) = \omega(\lambda)$. Thus, $U_k \Rightarrow U$ as $k \to \infty$. The theorem is thus proved. □

Two Tauberian theorems below are multidimensional analogues of the known Tauberian theorems due to J. Karamata (Karamata, 1930b; Karamata, 1931a; Karamata, 1931b). We introduce the variables t and τ which are related by the formula $t\tau = 1$.

THEOREM 1.3.3. *For a measure U on Γ, let there exist the Laplace transformation $\widetilde{U}(\lambda)$, $\lambda \in C$, and let $R(t)$ be a regularly varying at infinity function.*

(1) *If*

$$U(t\cdot)/R(t) \Rightarrow \Phi, \qquad t \to \infty, \tag{1.3.8}$$

then for any $\lambda \in C$

$$\widetilde{U}(\tau\lambda)/R(t) \to \psi(\lambda) < \infty, \qquad \tau \to 0, \tag{1.3.9}$$

where

$$\psi(\lambda) = \widetilde{\Phi}(\lambda), \qquad \forall\lambda \in C. \tag{1.3.10}$$

(2) *If* (1.3.9) *holds for some function* ψ, *then relations* (1.3.8) *and* (1.3.10) *hold as well for some measure* Φ *on* Γ.

PROOF. Let (1.3.8) be true. We observe that $\widetilde{U}(\tau\lambda)/R(t)$ is the Laplace transform of the measure $U(t\cdot)/R(t)$. Therefore, in view of Theorem 1.3.2, in order to prove (1.3.9) and (1.3.10) it suffices to show that $\widetilde{U}(\tau a)/R(t)$ are bounded for any $a \in C$ as $t \to \infty$. Let

$$D = \{x : x \in \mathbf{R}^n,\ |x| \le 1\}, \qquad B = (\partial D) \cap \Gamma.$$

Then

$$\widetilde{U}(\tau a) = \int_\Gamma e^{-(a\tau,x)} U(dx) = \sum_{k\ge 0} \int_{A_k} e^{-(a\tau,x)} U(dx),$$

where $A_k = A_k(t) = (2^k tD \setminus 2^{k-1}tD) \cap \Gamma$ for $k \in \mathbf{N}$ and $A_0 = (tD) \cap \Gamma$. It is easily seen that $(a\tau, x) \ge \tau|x| \inf_{b\in B}(a,b) \ge 2^{k-1}c$, where $c = \inf_{b\in B}(a,b) > 0$, for $x \in A_k$, $k \in \mathbf{N}$, because $a \in C = \operatorname{int}\Gamma^*$. As shown in Lemma 1.2.4, $\Phi(\partial D) = 0$, therefore,

$$U(tD)/R(t) \to \Phi(D), \qquad t \to \infty.$$

For $t \ge t_0$, let $U(tD) \le KR(t)$, where $\Phi(D) < K < \infty$, then for $t \ge t_0$

$$\widetilde{U}(\tau a) \le U(tD) + \sum_{k\in\mathbf{N}} e^{-c2^{k-1}} U(2^k tD) \le K(R(t) + \sum_{k\in\mathbf{N}} e^{-c2^{k-1}} R(2^k t)).$$

Hence it follows that $\widetilde{U}(\tau a)/R(t)$ are bounded (Feller, 1966, Section XIII.5, relation (5.11)).

If (1.3.9) is true, then (1.3.8) and (1.3.10) immediately follow from part 1 of Theorem 1.3.2. The proof of the theorem is thus complete. □

THEOREM 1.3.4. *For some function* $u(x) \ge 0$ *defined and measurable on* S, *let* $\hat{u}(\lambda) < \infty$ *for any* $\lambda \in C$.

(1) *If* $R(t)$ *is regularly varying at infinity,*

$$\hat{u}(\tau\lambda)/R(t) \to \psi(\lambda) < \infty, \quad t \to \infty, \quad \lambda \in C, \tag{1.3.11}$$

and $u(x) = f(x)g(x)$, *where* $f \in R_2(G)$ *(see Section* 1.1*), and* g *is monotone in the domain* G *(see Section* 1.2*), then for* $x \in G$

$$u(tx)t^n/R(t) \to \varphi(x) < \infty, \qquad t \to \infty, \tag{1.3.12}$$

and there exists a measure Φ *on* Γ *such that*

$$\Phi(dx) = \varphi(x)\,dx \text{ in } G \quad \text{and} \quad \widetilde{\Phi}(\lambda) = \psi(\lambda)\ \forall\lambda \in C. \tag{1.3.13}$$

(2) *If* $u \in R_2(\Gamma)$ *and* $\operatorname{ind} u > -n$, *then* (1.3.11) *and* (1.3.13) *are true with* $\varphi = H_a(u)$ *and* $R(t) = t^n u(ta)$ *for any* $a \in S$ *(see Section* 1.1*), and the relation* $\Phi(\partial\Gamma) = 0$ *certainly holds.*

(3) *Let* $R(t)$ *be a regularly varying at infinity function, and let* (1.3.11) *hold. If*

$$(u(tx_t) - u(tx))t^n / R(t) \to 0$$

as $t \to \infty$ *for any vectors* $x_t, x \in G$ *such that* $x_t \to x$, *then* (1.3.12) *and* (1.3.13) *hold for some function* φ *on* G *and some measure* Φ *on* Γ.

REMARK 1.3.1. If $\Phi(\partial\Gamma) = 0$ in Theorem 1.3.4, then by virtue of (1.3.13) the equality

$$\psi(\lambda) = \widehat{\varphi}(\lambda) < \infty$$

holds for any $\lambda \in C$.

PROOF. We set $U(dx) = u(x)\,dx$.

Let the hypotheses of part 1 be satisfied. Then by virtue of Theorem 1.3.3 there exists a measure Φ on Γ such that

$$\frac{U(t\cdot)}{R(t)} \Rightarrow \Phi, \qquad t \to \infty.$$

Thus,

$$\frac{U(tA)}{R(t)} = \int_A \frac{u(ty)t^n}{R(t)} dy \to \Phi(A), \qquad t \to \infty, \tag{1.3.14}$$

for an arbitrary set $A \in \mathfrak{A}$ such that $\Phi(\partial A) = 0$. We take an arbitrary vector $a \in S$ and set

$$h_t(y) = \frac{g(ty)t^n f(ta)}{R(t)}, \quad y \in G, \quad t > 0.$$

For any $t > 0$ the function $h_t(y) \in M^+(G)$ because $g \in M^+(G)$. Let us demonstrate that for any $x \in G$ there exists a constant $c_x < \infty$ such that for sufficiently large t

$$h_t(x) \le c_x. \tag{1.3.15}$$

If g does not decrease, then

$$\begin{aligned}\Phi(D) &= \lim_{t\to\infty} \int_D \frac{u(ty)t^n}{R(t)} dy \\ &\ge \limsup_{t\to\infty} \int_{D_1} \frac{f(ty)}{f(ta)} h_t(y)\,dy \ge |D_1| \inf_{y\in D} \frac{f(ty)}{f(ta)} h_t(x),\end{aligned} \tag{1.3.16}$$

for an arbitrary ball D with centre at the point tx such that $D \subseteq G$ and $\Phi(\partial D) = 0$, where $D_1 = \{y \colon y \in D,\ y \overset{T}{>} x\}$. Let $H_a(f) = \varphi_1$. Then by virtue of Theorem 1.1.2, since the function φ_1 is continuous and positive, there exist $b > 0$ and $t_0 > 0$ such that

$$\inf_{y\in D} \frac{f(ty)}{f(ta)} \ge b > 0$$

for all $t \geq t_0$; hence it follows that (1.3.15) holds for $t \geq t_0$ with $c_x = \frac{\Phi(D)}{b|D_1|}$. If g does not increase, then the same reasoning yields

$$\Phi(D) \geq |D_2| \inf_{y \in D} \frac{f(ty)}{f(ta)} h_t(x),$$

where $D_2 = \{y\colon y \in D, y \overset{T}{<} x\}$, therefore, (1.3.15) holds for $t \geq t_0$ with $c_x = \frac{\Phi(D)}{b|D_2|}$.

By virtue of Theorem 1.2.1, the set of functions $\{h_t(x),\ t \geq t_0\}$ is weakly relatively compact. As $t_k \uparrow \infty$, let

$$h_{t_k} \Rightarrow h, \qquad k \to \infty. \tag{1.3.17}$$

Let us demonstrate that the function $\varphi_1(x)h(x)$ is the density of the measure Φ with respect to the Lebesgue measure in G. Let the set $A \ll \Gamma$, $A \in \mathfrak{A}$. Then there exist $\varepsilon > 0$ and $\lambda > 0$ such that $\varepsilon a \overset{\Gamma}{<} x \overset{\Gamma}{<} \lambda a$ for any $x \in A$. It is easy to see, indeed, that if for arbitrary $\varepsilon > 0$ there exists $x \in A$ such that $x - a\varepsilon \in E = \mathbf{R}^n \setminus G$, then there exist sequences $x_k \to x \in \bar{A}$ and $\varepsilon_k \to 0$ as $k \to \infty$ such that $x_k - a\varepsilon_k \in E$. By passing to the limit as $k \to \infty$, we find that $x \in E$ because E is closed, which is impossible. Similarly, if $a\lambda - x \in E$ for any $\lambda > 0$ and some $x \in A$ depending on λ, then there exist sequences $\lambda_k \to \infty$ and $x_k \to x \in \bar{A}$ such that $a\lambda_k - x_k \in E$ or $a - x_k/\lambda_k \in E$. If k tends to infinity, we see that $a \in E$, which is impossible. Thus, there exist vectors $u, v \in G$ such that $u < x < v$ for any $x \in A$. By virtue of Theorem 1.1.2, there exist $t_1 > 0$ and a constant $c > 0$ such that

$$\frac{f(tx)}{f(ta)} \leq c$$

for $t \geq t_1$ and $x \in \bar{A}$. In view of relation (1.3.15) and the monotonicity of h_t, $h_t(x) \leq \max(c_u, c_v)$ for sufficiently large $t \geq t_2$ and $x \in A$. Therefore,

$$\frac{u(tx)t^n}{R(t)} = \frac{f(tx)}{f(ta)} h_t(x) \leq c \max(c_u, c_v) \tag{1.3.18}$$

for $t \geq \max(t_1, t_2)$ and $x \in A$ By virtue of the Lebesgue theorem, from (1.3.17) and (1.3.18) we obtain

$$\int_A \frac{u(t_k)t_k^n}{R(t_k)} dx \to \int_A \varphi_1(x)h(x)dx, \qquad k \to \infty. \tag{1.3.19}$$

We assume that $\Phi(\partial A) = 0$. Then, along with (1.3.19), (1.3.14) holds, and therefore,

$$\Phi(A) = \int_A \varphi_1(x)h(x)\,dx.$$

According to the last relation, the function h is uniquely defined by Φ up to a set of Lebesgue measure zero. But $h \in M^+(G)$ by (1.3.17), that is, $h(x) = h(x_+)$, so h is uniquely defined by Φ. It is easy to see, indeed, that if $h_1, h_2 \in M^+(G)$ and $h_1(x) = h_2(x)$ almost everywhere in G, then for an arbitrary $x \in G$ there exists a sequence $x_k \downarrow x$ as $k \to \infty$ such that $h_1(x_k) = h_2(x_k)$. By passing to the limit in the last equality as $k \to \infty$, with the use of Lemma 1.2.2 we conclude that $h_1(x_+) = h_2(x_+)$ or $h_1(x) = h_2(x)$. Therefore, keeping in mind that the limit h does not depend on the sequence $(t_k,\ k \in \mathbf{N})$, we see that $h_t \Rightarrow h$ as $t \to \infty$, that is,

$$g(tx)r(t) \to h(x), \qquad t \to \infty, \tag{1.3.20}$$

for an arbitrary point $x \in G$ of continuity of h, where $r(t) = t^n f(ta)/R(t)$ is regularly varying at infinity. For any $x \in G$, $\lambda > 0$, $\mu > 0$ such that h is continuous at the points λx and μx, we see that

$$\begin{aligned} h(\lambda x) &= \lim_{t\to\infty} g(t\lambda x)r(t) = \lim_{b\to\infty} g(b\mu x) r\left(b\frac{\mu}{\lambda}\right) \\ &= \left(\frac{\mu}{\lambda}\right)^{\alpha} \lim_{b\to\infty} g(b\mu x)r(b) = \left(\frac{\mu}{\lambda}\right)^{\alpha} h(\mu x), \end{aligned}$$

or

$$h(\lambda x) = \left(\frac{\mu}{\lambda}\right)^{\alpha} h(\mu x), \tag{1.3.21}$$

where $\alpha = \operatorname{ind} r$. We set $\sigma = \{y \colon y \in G,\ y \text{ is a continuity point of } h\}$. Then, by virtue of (1.3.21) and Lemma 1.2.2, we obtain

$$\begin{aligned} h(x_+) &= \lim_{\lambda\downarrow 1, \lambda x\in\sigma} h(\lambda x) = \mu^{\alpha} h(\mu x), \\ h(x_-) &= \lim_{\lambda\uparrow 1, \lambda x\in\sigma} h(\lambda x) = \mu^{\alpha} h(\mu x). \end{aligned}$$

By Lemma 1.2.2, h is continuous at x. Thus, (1.3.20) holds for all $x \in G$. Let $\varphi = \varphi_1 h$. Then from (1.3.20) and the fact that $f \in R_2(G)$ it follows that

$$\frac{t^n u(tx)}{R(t)} \to \varphi(x), \qquad t \to \infty,$$

for $x \in G$, which completes the proof of part 1 of the theorem.

Let us prove the second part of the theorem. Let $u \in R_2(\Gamma)$. For some vector $a \in S$, we set

$$R(t) = t^n u(ta), \quad \varphi = H_a(u), \quad L(x) = u(x)/\varphi(x), \quad \rho = \operatorname{ind} \varphi.$$

By virtue of Theorem 1.1.2, $L \in T_2(\Gamma)$. We observe that for $x \in S$

$$0 < \varphi(x) = |x|^{\rho} \varphi\left(\frac{x}{|x|}\right) \le |x|^{\rho} \max_{b\in B} \varphi(b).$$

Therefore, for $\varepsilon < n + \rho$ the function $|x|^{-\varepsilon}\varphi(x)$ is locally integrable on S with respect to the Lebesgue measure, because $\rho > -n$. For each such $\varepsilon > 0$ and arbitrary $A \in \mathfrak{A}$, $A \subseteq \Gamma$, we see that

$$\frac{U(tA)}{R(t)} = \int_A \frac{u(tx)}{v(t)} dx = \int_A \varphi(x) \frac{L(tx)}{L(ta)} dx,$$

where $v(t) = u(ta)$. Let $b > 0$ be chosen so that for any compact $K \subseteq \{x: x \in S, |x| \geq b\}$ the inequalities

$$0 < \inf_{x \in K} L(x) \leq \sup_{x \in K} L(x) < \infty$$

hold (see Lemma 1.1.2). We set $D = \{x: x \in \mathbf{R}^n,\ |x| \leq 1\}$. Then

$$\begin{aligned}\frac{U(tA)}{R(t)} &= \int_A \varphi(x)\frac{L(tx)}{L(ta)}dx \leq \frac{U(bD)}{R(t)} + \frac{\sup_{y \in tA, |y| \geq b} |y|^{\varepsilon} L(y)}{t^{\varepsilon} L(ta)} \int_A \varphi(x)|x|^{-\varepsilon} dx \\ &= o(1) + (1 + o(1)) \int_A \varphi(x)|x|^{-\varepsilon} dx, \qquad t \to \infty,\end{aligned}$$

by virtue of Lemma 1.1.3 and the fact that $R(t) \to \infty$ as $t \to \infty$ (because $\operatorname{ind} R > 0$). From the last bounds it follows that

$$\limsup_{t \to \infty} \frac{U(tA)}{R(t)} \leq \int_A \varphi(x)|x|^{-\varepsilon} dx$$

for any $\varepsilon \in (0, n + \rho)$. By virtue of the Lebesgue theorem,

$$\int_A \varphi(x)|x|^{-\varepsilon} dx \to \int_A \varphi(x)\, dx, \qquad \varepsilon \to \infty.$$

Therefore,

$$\limsup_{t \to \infty} \frac{U(tA)}{R(t)} \leq \int_A \varphi(x)\, dx. \tag{1.3.22}$$

Estimating $U(tA)/R(t)$ from below, we obtain

$$\frac{U(tA)}{R(t)} \geq \frac{U(bD)}{R(t)} + \frac{\inf_{y \in tA, |y| \geq b} |y|^{-\varepsilon} L(y)}{t^{-\varepsilon} L(ta)} \int_A \varphi(x)|x|^{\varepsilon} dx$$

for any $\varepsilon > 0$; therefore,

$$\liminf_{t \to \infty} \frac{U(tA)}{R(t)} \geq \int_A \varphi(x)\, dx. \tag{1.3.23}$$

From (1.3.22) and (1.3.23) it follows that

$$\frac{U(tA)}{R(t)} \to \int_A \varphi(x)\, dx, \qquad t \to \infty.$$

Formulas (1.3.11) and (1.3.13) immediately follow from the last relation and Theorem 1.3.3. Part 2 of the theorem is thus proved.

Let the hypotheses of the third part of the theorem be satisfied. Let us demonstrate that

$$\lim_{\substack{|x-y| \to 0 \\ x, y \in K}} \limsup_{i \to \infty} \left| \frac{u(ty) - u(tx)}{v(t)} \right| = 0 \tag{1.3.24}$$

for any compact $K \subseteq G$, where $v(t) = t^{-n}R(t)$. Assume the contrary: let there exist sequences $x_k, y_k \in K$, $t_k \to \infty$, $|x_k - y_k| \to 0$ as $k \to \infty$, and $\varepsilon > 0$ such that

$$\left|\frac{u(t_k y_k) - u(t_k x_k)}{v(t)}\right| > \varepsilon. \tag{1.3.25}$$

Without loss of generality we assume that $x_k \to x \in K$ as $k \to \infty$ and $0 < t_1 < t_2 < \cdots < t_k < \cdots$ Then $y_k \to x$ as $k \to \infty$. Let $a(t) = x_k, b(t) = y_k$ for $t \in [t_k, t_{k+1})$, then $a(t) \to x$, $b(t) \to x$ as $t \to \infty$, therefore,

$$\frac{u(ta(t)) - u(tx)}{v(t)} \to 0, \qquad \frac{u(tb(t)) - u(tx)}{v(t)} \to 0$$

as $t \to \infty$; hence

$$\frac{u(ta(t)) - u(tb(t))}{v(t)} \to 0, \qquad t \to \infty.$$

For $t \ge t_0$, let

$$\left|\frac{u(ta(t)) - u(tb(t))}{v(t)}\right| < \varepsilon.$$

The last relation for $t = t_k, t_k \ge t_0$ contradicts (1.3.25). Thus, (1.3.24) is true. By virtue of Theorem 1.3.3,

$$U(t\cdot)/R(t) \Rightarrow \Phi, \qquad t \to \infty, \tag{1.3.26}$$

for some measure Φ on Γ, and

$$\widetilde{\Phi}(\lambda) = \psi(\lambda) < \infty \qquad \forall \lambda \in C. \tag{1.3.27}$$

We fix an arbitrary $x \in G$ and take $(r_k > 0,\ k \in \mathbf{N})$ so that $A_k = \{y\colon y \in \mathbf{R}^n,\ |y - x| \le r_k\} \subseteq G$, $\Phi(\partial A_k) = 0$. Then

$$\begin{aligned}\frac{u(tx)}{v(t)} &= |A_k|^{-1}\int_{A_k}\frac{u(tx)}{v(t)}dy = |A_k|^{-1}\left(\frac{U(tA_k)}{R(t)} - \int_{A_k}\frac{u(ty) - u(tx)}{v(t)}dy\right)\\ &\le |A_k|^{-1}\frac{U(tA_k)}{R(t)} + \sup_{y\in A_k}\frac{|u(ty) - u(tx)|}{v(t)}.\end{aligned} \tag{1.3.28}$$

From (1.3.24) and (1.3.28) it follows, first, that $u(tx)/v(t)$ is bounded for sufficiently large t, and second,

$$\limsup_{t\to\infty}\frac{u(tx)}{v(t)} \le \frac{\Phi(A_k)}{|A_k|} + \limsup_{t\to\infty}\sup_{y\in A_k}\frac{|u(ty) - u(tx)|}{v(t)}.$$

Taking into account (1.3.24) from the last inequality we arrive at

$$\limsup\frac{u(tx)}{v(t)} \le \liminf_{k\to\infty}\frac{\Phi(A_k)}{|A_k|} < \infty.$$

Reasoning as above but estimating from below, we see that

$$\lim_{t\to\infty}\frac{u(tx)}{v(t)} = \lim\frac{\Phi(A_k)}{|A_k|} \equiv \varphi(x) < \infty.$$

Thus,

$$\frac{u(tx)}{v(t)} \to \varphi(x) < \infty \quad (t \to \infty). \tag{1.3.29}$$

Let us demonstrate that φ is continuous in x. From (1.3.24) it follows that for an arbitrary $\varepsilon > 0$ there exist $\delta > 0$ and $b > 0$ such that

$$\frac{|u(ty) - u(tx)|}{v(t)} < \varepsilon$$

for $|y - x| < \delta$ and $t \geq b$. By passing in the last inequality to the limit as $t \to \infty$, we find that $|\varphi(y) - \varphi(x)| \leq \varepsilon$. Next, let us show that (1.3.29) holds uniformly in $x \in K$ for an arbitrary compact $K \subseteq G$. We assume the contrary that there are $x_k \in K$, $x_k \to x \in K$, $t_k \uparrow \infty$ as $k \to \infty$ such that

$$\left| \frac{u(t_k x_k)}{v(t_k)} - \varphi(x_k) \right| > \varepsilon_0 > 0. \tag{1.3.30}$$

But by (1.3.24), (1.3.29) and continuity of φ there exists m such that for $k \geq m$

$$\frac{|u(t_k x_k) - u(t_k x)|}{v(t_k)} < \varepsilon_0/3,$$
$$|\varphi(x_k) - \varphi(x)| < \varepsilon_0/3,$$
$$\left| \frac{u(t_k x)}{v(t_k)} - \varphi(x) \right| < \varepsilon_0/3,$$

which contradicts (1.3.30). Thus,

$$\int_A \frac{u(ty)}{v(t)} dy \to \int_A \varphi(y)\, dy, \qquad t \to \infty,$$

for any set $A \in \mathfrak{A}$, $A \ll \Gamma$. If A is chosen so that $\Phi(\partial A) = 0$, then from (1.3.26) we obtain

$$\int_A \frac{u(ty)}{v(t)} dy \to \Phi(A), \qquad t \to \infty.$$

From the last two relations it follows that $\Phi(dx) = \varphi(x)\, dx$ in G. Taking into account (1.3.27), hence we obtain (1.3.13). The theorem is proved. □

1.4. Weakly oscillating functions

As a possible generalisation of regularly varying functions of many variables considered in Section 1.1, one can consider weakly oscillating functions introduced in (Yakymiv, 1988). We use them to prove the multidimensional Tauberian comparison theorem in Section 1.5 and in probabilistic framework in Chapters 2–5. In order to study weakly oscillating functions, we need to introduce sequences of so-called asymptotically continuous functions and analyse their properties.

Let D be a separable locally compact metric space. Let $r(a, b)$ denote the distance between elements a and b of D. Everywhere in this section $(h_m(x),\ m \in \mathbf{N})$ is a sequence of complex-valued functions defined on D.

DEFINITION 1.4.1. A sequence $(h_m(x),\ m \in \mathbf{N})$ is said to be *asymptotically continuous* in D, if for an arbitrary $x \in D$

$$h_m(y) - h_m(x) \to 0, \quad m \to \infty, \quad y \to x. \tag{1.4.1}$$

LEMMA 1.4.1. *Let a sequence $(h_m(x),\ m \in \mathbf{N})$ be asymptotically continuous in D. Then*

$$h_m(x) - h_m(y) \to 0$$

for any compact $K \subseteq D$ and for $x, y \in K$ such that $r(x, y) \to 0$ as $m \to \infty$.

PROOF. We assume the contrary that there exists an unbounded set $L \subseteq N$, sequences $\delta_m \downarrow 0$, $x_m, y_m \in K$, $r(x_m, y_m) \le \delta_m$ and $\varepsilon > 0$ such that for all $m \in L$

$$|h_m(y_m) - h_m(x_m)| \ge \varepsilon.$$

Without loss of generality we assume that $x_m, y_m \to x \in K$ as $m \to \infty$, $m \in L$. Then

$$|h_m(y_m) - h_m(x_m)| \le |h_m(y_m) - h_m(x)| + |h_m(x_m) - h_m(x)| \to 0$$

as $m \to \infty$, $m \in L$, because the sequence $(h_m(x))$ is asymptotically continuous. This contradiction proves the lemma. □

THEOREM 1.4.1. *Let a sequence $(h_m(x))$ be asymptotically continuous in D. In order for it to be pre-compact in D in the pointwise convergence topology, it is necessary and sufficient that for any $x \in D$*

$$\limsup_{m\to\infty} |h_m(x)| < \infty. \tag{1.4.2}$$

PROOF. It is clear that condition (1.4.2) is necessary. Let us prove its sufficiency. Let (1.4.2) hold and $D' = \{x_1, x_2, x_3, \dots\}$ be a countable everywhere dense set. With the use of Cantor's technique we construct an unbounded subset $L \subseteq \mathbf{N}$ such that for all $x \in D'$

$$h_m(x) \to h(x), \quad m \to \infty, \quad m \in L, \tag{1.4.3}$$

for some function $h(x)$ on D', where $|h(x)| < \infty$. It is easily seen that $h(x)$ is uniformly continuous on the set $M = K \cap D'$ for any compact $K \subseteq D$. By virtue of Lemma 1.4.1, for any $\varepsilon > 0$ there exist $l \in \mathbf{N}$ and $\delta > 0$ such that

$$|h_m(x) - h_m(y)| < \varepsilon \tag{1.4.4}$$

for any $x, y \in K$, $r(x, y) < \delta$, $m \in \mathbf{N}$, $m > l$. Now let $x, y \in M$. If m grows without bound in (1.4.4), recalling (1.4.3) we see that

$$|h(x) - h(y)| \le \varepsilon,$$

which implies the uniform continuity of the function h on M. Further, for any $x \in D \setminus D'$ there exists

$$\lim_{y\to x, y\in D'} h(y) \stackrel{\text{def}}{=} h(x). \tag{1.4.5}$$

To see this, let $h(x_m) \to c$ for some sequence $x_m \in D'$, $x_m \to x$, and let a sequence $y_m \in D'$ be such that $y_m \to x$ as $m \to \infty$. We choose $\delta > 0$ in such a way that the set

$$K(x, \delta) = \{y: y \in D, r(x, y) \le \delta\}$$

is a compact in D. Then from the uniform continuity of the set h on the set $K(x, \delta) \cap D'$ it follows that

$$|h(y_m) - c| \le |h(y_m) - h(x_m)| + |h(x_m) - c| \to 0, \qquad m \to \infty,$$

which implies (1.4.5). It remains to show that (1.4.3) holds for all $x \in D \setminus D'$. We choose arbitrary $x \in D \setminus D'$ and $\varepsilon > 0$. For them, we find $l \in \mathbf{N}$ and $\delta > 0$ such that

$$|h_m(x) - h_m(a)| < \varepsilon/3$$

for all $m > l$ and $a \in K(x, \delta)$. By (1.4.5), there exists $a \in D' \cap K(x, \delta)$ such that

$$|h(a) - h(x)| < \varepsilon/3.$$

For this a, we take $k > l$ such that

$$|h_m(a) - h(a)| < \varepsilon/3$$

for all $m > k, m \in L$. Then

$$|h_m(x) - h(x)| \le |h_m(x) - h_m(a)| + |h_m(a) - h(a)| + |h(a) - h(x)| < \varepsilon$$

for any $m \in L$, $m > k$, which is the desired result. The theorem is proved. □

THEOREM 1.4.2. *We assume that a sequence* $(h_m(x),\ m \in \mathbf{N})$ *is asymptotically continuous in* D *and that*

$$h_m(x) \to h(x), \qquad |h(x)| < \infty, \tag{1.4.6}$$

for all $x \in D$ *as* $m \to \infty$. *Then the function* $h(x)$ *is continuous in* D *and relation* (1.4.6) *holds uniformly in* $x \in K$ *for any compact* $K \subseteq D$.

PROOF. The continuity of $h(x)$ follows from relations (1.4.1) and (1.4.6). We assume that there exist a compact $K \subseteq D$, $\varepsilon > 0$, a sequence $x_m \in K$, and an unbounded set $L \subseteq N$ such that

$$|h_m(x_m) - h(x_m)| > \varepsilon \tag{1.4.7}$$

for any $m \in L$. Without loss of generality we take $x_m \to x \in K$. Since

$$|h_m(x_m) - h(x_m)| \le |h_m(x_m) - h_m(x)| + |h_m(x) - h(x)| + |h(x) - h(x_m)|,$$

in view of the asymptotic continuity of h_m, relation (1.4.6) and the continuity of $h(x)$ we obtain

$$|h_m(x_m) - h(x_m)| \to 0, \quad m \to \infty, \quad m \in L,$$

which contradicts (1.4.7). The theorem is proved. □

THEOREM 1.4.3. *Let D be connected and $(h_m(x))$ be asymptotically continuous. Then*

$$\limsup_{m\to\infty} \sup_{x,y\in K} |h_m(x) - h_m(y)| < \infty \tag{1.4.8}$$

for any compact $K \subseteq D$.

PROOF. We take arbitrary $a, b \in D$. Let $\{x(t), t \in [0,1]\}$ be a continuous curve which joins the points a and b, $x(0) = a$, $x(1) = b$. We set

$$T = \{t : t \in [0,1],\ \limsup_{m\to\infty} |h_m(a) - h_m(x(t))| = \infty\}.$$

Let us show that $T = \varnothing$. Let the contrary be true, that is, $T \neq \varnothing$, and let $v = \inf T$. It is clear that $v > 0$. In view of continuity of the curve $x(t)$ and asymptotic continuity of (h_m), we obtain

$$h_m(x(s)) - h_m(x(t)) \to 0, \quad m \to \infty, \quad s, t \to v.$$

Therefore, there are $k \in \mathbf{N}$ and $\delta \in (0, v)$ such that

$$|h_m(x(s)) - h_m(x(t))| < 1 \tag{1.4.9}$$

for any $s, t \in [v - \delta, v + \delta]$, $m \in \mathbf{N}$, $m > k$. We take arbitrary $t \in [v - \delta, v)$ and $s \in [v, v + \delta] \cap T$. Then by (1.4.9)

$$\limsup_{m\to\infty} |h_m(a) - h_m(x(s))| < 1 + \limsup_{m\to\infty} |h_m(a) - h_m(x(t))| < \infty,$$

which contradicts our choice of s, $s \in T$.

Thus, $T = \varnothing$. Hence

$$\limsup_{m\to\infty} |h_m(a) - h_m(b)| < \infty \tag{1.4.10}$$

for any $a, b \in D$. Assume that (1.4.8) does not hold, that is, for some compact $K \subseteq D$ there exists an unbounded set $L \subseteq \mathbf{N}$ and sequences $x_m, y_m \in K$ such that

$$|h_m(x_m) - h_m(y_m)| \to \infty, \quad m \to \infty, \quad m \in L.$$

Without loss of generality we assume that $x_m \to a \in K$, $y_m \to b \in K$ as $m \to \infty$, $m \in L$. By virtue of the asymptotic continuity of (h_m), hence it follows that

$$|h_m(a) - h_m(b)| \to \infty, \quad m \to \infty, \quad m \in L,$$

which contradicts (1.4.10). The theorem is proved. □

DEFINITION 1.4.2. A family of functions $(h_t(x),\ t \geq t_0)$ is said to be *asymptotically continuous* in D as $t \to \infty$ if for any $x \in D$

$$h_t(y) - h_t(x) \to 0, \quad t \to \infty, \quad y \to x.$$

REMARK 1.4.1. It is clear that, in view of Definition 1.4.2, Lemma 1.4.1, and Theorems 1.4.1, 1.4.2, 1.4.3 remain true after changing $m \in \mathbf{N}$ for 'continuous' parameter $t \geq t_0$.

As above, let Γ be a cone in $\mathbf{R}^n$ with apex at zero, $S = \Gamma \setminus \{0\}$, and let t be a non-negative variable.

DEFINITION 1.4.3. A non-negative function $f(x)$ defined for $x \in \Gamma$, $|x| \geq a \geq 0$, is called *feebly oscillating* (at infinity in Γ) if there exists a vector $e \in S$ such that $f(te) > 0$ for all sufficiently large t, and for arbitrary $x \in S$, as $t \to \infty$,

$$f(tx_t) - f(tx) = o(f(te)), \tag{1.4.11}$$

provided that $x_t \in S$, $x_t \to x$.

Unless otherwise stated, in what follows we assume that S is connected. For illustrative purpose, we give a series of examples of feebly oscillating functions for $n = 1$, (here $\Gamma = \mathbf{R}_1^+ \equiv \{t, t \geq 0\}$).

(1) Let a function $f(x) > 0$ be defined and differentiable for $x \geq a \geq 0$, and for some real α, β let

$$\alpha f(x) \leq x f'(x) \leq \beta f(x).$$

Then f is feebly oscillating at infinity (such functions were introduced in (Keldysh, 1973) and then used in Tauberian theorems, see, e.g., (Selander, 1963)).

(2) For all $c > 1$, let there exist constants α, β, and $v > 0$ such that

$$c^{-1}(x/y)^{\alpha} \leq f(x)/f(y) \leq c(x/y)^{\beta}$$

for any $x > y > v$. Then f is feebly oscillating at infinity. For monotone functions, one of the last inequalities is certainly true (such conditions on monotone functions f were used in Tauberian theorems proved in (Belogrud, 1974; Matsaev, Palant, 1977; Sultanaev, 1974); see also the monograph (Kostyuchenko, Sargsyan, 1979)).

(3) Let a function $f(x) > 0$ do not decrease, and for all $\lambda > 1$ let

$$\limsup_{t \to \infty} \frac{f(\lambda t)}{f(t)} \equiv \varphi(\lambda) < \infty,$$

while $\varphi(1_+) = 1$. Then f is feebly oscillating at infinity (such conditions of functions f were used in Tauberian theorems given in (Stadtmüller, Trautner, 1979; Stadtmüller, Trautner, 1981; Stadtmüller, 1983)). Similar example can be formulated for non-increasing functions.

(4) Let f be a function regularly varying at infinity, that is, be positive, measurable, and

$$f(\lambda t)/f(t) \to \varphi(\lambda) > 0, \qquad \varphi(\lambda) < \infty,$$

for any $\lambda > 0$ as $t \to \infty$. Then f is feebly oscillating at infinity. Such functions were first introduced by J. Karamata (Karamata, 1930a; Karamata, 1930b; Karamata, 1931a) in connection with establishing extensions of a known Tauberian theorem due to Hardy and Littlewood (Hardy, Littlewood, 1914). Now regularly varying functions enjoy wide application, see (Bingham *et al.*, 1987).

We observe that feebly oscillating functions may oscillate between two power functions. For example, the function

$$f(x) = x^{\sin \ln \ln x}$$

oscillates between x^{-1} and x but very slowly, so remains in the class of feebly oscillating functions. In passing it should be mentioned that the regularly varying functions oscillate between $x^{\rho-\varepsilon}$ and $x^{\rho+\varepsilon}$ for any fixed $\varepsilon > 0$, where ρ is the index of regularly varying function (Seneta, 1976; Bingham *et al.*, 1987). The functions in the above examples 1–3 are, generally speaking, not regularly varying, as seen from the function

$$f(x) = x^{2+\sin \ln \ln x}.$$

For feebly oscillating functions, the following theorem is true.

THEOREM 1.4.4. *Let a function f be feebly oscillating at infinity in Γ. Then for any compact $K \subseteq S$*

$$\limsup_{t\to\infty} \sup_{x\in K} \frac{f(tx)}{f(te)} < \infty, \tag{1.4.12}$$

and the limit

$$\lim_{\substack{\delta\to 0\\ t\to\infty}} \sup_{\substack{x,y\in K\\ |x-y|\le\delta}} \frac{|f(ty) - f(tx)|}{f(te)} = 0 \tag{1.4.13}$$

exists.

PROOF. Let $f(te) > 0$ for $t \geq t_0$. For $x \in S$ and $t \geq t_0$, we set

$$h_t(x) = \frac{f(tx)}{f(te)}$$

(without loss of generality, we let f be defined in the whole S). Let us demonstrate that the family of functions $(h_t(x),\ t \geq t_0)$ is asymptotically continuous in S as $t \to \infty$ (see Definition 1.4.2). We assume the contrary: let there exist sequences $t_m \to \infty$, $x_m \in S$, $x_m \to x \in S$ as $m \to \infty$ and $\varepsilon > 0$ such that for all $m \in \mathbf{N}$

$$\frac{|f(t_m x_m) - f(t_m x)|}{f(t_m e)} > \varepsilon. \tag{1.4.14}$$

We also assume that $t_0 < t_1 < t_2 < \ldots$. We set $x(t) = x_m$ for $t \in [t_m, t_{m+1})$. It is clear that $x(t) \to x$ as $t \to \infty$. By (1.4.11),

$$f(tx(t)) - f(tx) = o(f(te)), \qquad t \to \infty.$$

Setting $t = t_m$ in the last equation, we obtain

$$f(t_m x_m) - f(t_m x) = o(f(t_m e)), \qquad m \to \infty,$$

which contradicts (1.4.14). Thus, the family of functions $(h_t(x),\ t \geq t_0)$ is asymptotically continuous in S. Therefore, from Theorem 1.4.3 (see Remark 1.4.1) it follows that for any compact $K \subseteq S$

$$\lim_{t\to\infty} \sup_{x,y\in K} \frac{|f(tx) - f(ty)|}{f(te)} < \infty. \tag{1.4.15}$$

Without loss of generality, we assume that $e \in K$ (the compact K, if needed, may be extended by one point). It is clear that

$$\frac{f(tx)}{f(te)} \leq \frac{|f(tx) - f(te)|}{f(te)} + 1. \tag{1.4.16}$$

Now (1.4.12) follows from (1.4.15), (1.4.16) because $e \in K$. Relation (1.4.13) follows from Lemma 1.4.1 and asymptotic continuity of the family of functions $(h_t(x),\ t \geq t_0)$. The theorem is proved. □

REMARK 1.4.2. From (1.4.11) it follows that for all $\lambda > 0$, $x_t, x \in S$, $x_t \to x$ as $t \to \infty$

$$\frac{f(tx_t) - f(tx)}{f(t\lambda e)} \to 0.$$

Ii is easily seen indeed that

$$\frac{f(tx_t) - f(tx)}{f(t\lambda e)} = \frac{f((t\lambda)(x_t\lambda^{-1})) - f((t\lambda)(x\lambda^{-1}))}{f(t\lambda e)} \to 0$$

as $t \to \infty$ by (1.4.11).

COROLLARY 1.4.1. *If f is feebly oscillating ((1.4.11) holds), then for all* $0 < a < b < \infty$

$$0 < \liminf_{t\to\infty} \inf_{\lambda\in[a,b]} \frac{f(t\lambda e)}{f(te)}.$$

It is easily seen indeed that for $\tau = t\lambda$

$$\frac{f(t\lambda e)}{f(te)} = \frac{f(\tau e)}{f(\tau\lambda^{-1}e)} = \left(\frac{f(\tau(\lambda^{-1}e))}{f(\tau e)}\right)^{-1} \geq c > 0$$

for some $c > 0$ and all t large enough by (1.4.12).

If S is not connected and f feebly oscillates in Γ, then (1.4.12) may be broken.

For example, let $\Gamma = \mathbf{R}^1$, $f(x) = 1, x \leq -1$; $f(x) = \ln x, x \geq 1$. It is easy to see that f obeys (1.4.11) with $e = -1$, that is, f is feebly oscillating, but as $t \to \infty$

$$\frac{f(t)}{f(te)} = \ln t \to \infty,$$

in other words, (1.4.12) does not hold.

Let us give a definition of weakly oscillating functions.

DEFINITION 1.4.4. A function $f(x)$, which is defined and positive for $x \in \Gamma$, $|x| \geq a \geq 0$, is said to be weakly oscillating (at infinity in Γ) if

$$f(tx_t)/f(tx) \to 1 \tag{1.4.17}$$

for all $x_t, x \in S$, $x_t \to x$, as $t \to \infty$.

REMARK 1.4.3. In the definition of a weakly oscillating function it suffices to consider (1.4.17) to be true for all $x \in S$, $|x| = 1$.

It is easily seen indeed that if $|x| = 1$ and (1.4.17) holds, then for any positive v

$$f(tvx_t)/f(tvx) = f((tv)x_t)/f((tv)x) \to 1, \qquad t \to \infty.$$

THEOREM 1.4.5. *Let a function f be weakly oscillating at infinity. Then*

(1) *the function f is feebly oscillating and* (1.4.11) *holds for any vector $e \in S$.*

(2) *the relations*

$$0 < \liminf_{t\to\infty} \inf_{x,y\in K} \frac{f(tx)}{f(ty)} \leq \limsup_{t\to\infty} \sup_{x,y\in K} \frac{f(tx)}{f(ty)} < \infty \tag{1.4.18}$$

hold for any compact $K \subseteq S$; the limit

$$\lim_{\substack{\delta\to 0\\ t\to\infty}} \sup_{\substack{x,y\in K\\ |x-y|\leq\delta}} \left| \frac{f(ty)}{f(tx)} - 1 \right| = 0 \tag{1.4.19}$$

exists.

PROOF. If (1.4.19) does not hold, then there exist $\varepsilon > 0$ and sequences $t_k \uparrow \infty$, x_k, y_k, $x_k \to x$, $y_k \to x$ as $k \to \infty$, such that

$$|f(t_k y_k)/f(t_k x_k) - 1| \geq \varepsilon > 0.$$

If we now set $a(t) = x_k$ and $b(t) = y_k$ for $t \in [t_k, t_{k+1})$, then we see that for $t = t_k$

$$|f(tb(t))/f(ta(t)) - 1| \geq \varepsilon > 0.$$

But in view of (1.4.17)

$$\frac{f(ta(t))}{f(tb(t))} = \frac{f(ta(t))}{f(tx)} \frac{f(tx)}{f(tb(t))} \to 1, \qquad t \to \infty.$$

This contradiction proves (1.4.19).

We take arbitrary vectors $x, y \in S$. According to (1.4.19), there are $\varepsilon > 0$ and $t_0 > 0$ such that

$$|f(ta)/f(tb) - 1| \leq 1/2$$

for all $a, b \in I$, $|a - b| \leq \varepsilon$, $t \geq t_0$, where $I = \{z: z = x(t),\ t \in [0,1]\}$ is a continuous curve which joins the points x and y while $x(0) = x$, $x(1) = y$, $I \subseteq S$. We choose $m \in \mathbf{N}$ so that $w(1/m) \leq \varepsilon$, and set

$$x_1 = x(1/m), \quad x_2 = x(2/m), \quad \dots, \quad x_{m-1} = x((m-1)/m),$$

where $w(\delta)$ is the modulus of continuity of the curve $x(t)$. Then

$$2^m \geq \frac{f(tx)}{f(tx_1)} \cdots \frac{f(tx_{m-1})}{f(ty)} \geq \left(\frac{1}{2}\right)^m . \tag{1.4.20}$$

We assume that (1.4.18) does not hold. Then there exist sequences $t_k \to \infty$, $x_k, y_k \in K$, such that as $k \to \infty$

$$f(t_k x_k)/f(t_k y_k) \to \infty.$$

Without loss of generality we assume that $x_k \to x \in K$, $y_k \to y \in K$ as $k \to \infty$. By virtue of (1.4.19),

$$\frac{f(t_k x_k)}{f(t_k x)} \to 1, \qquad \frac{f(t_k y_k)}{f(t_k y)} \to 1$$

as $k \to \infty$. Therefore,

$$\frac{f(t_k x)}{f(t_k y)} = \frac{f(t_k x_k)}{f(t_k y_k)} \frac{f(t_k x)}{f(t_k x_k)} \frac{f(t_k y_k)}{f(t_k y)} \to \infty$$

as $k \to \infty$, which contradicts (1.4.20). Thus, (1.4.18) is proved.

Let $x_t, x, e \in S$, $x_t \to x$ as $t \to \infty$. By (1.4.18), there exist $c > 0$ and $t_0 > 0$ such that for $t \geq t_0$

$$f(tx)/f(te) \geq c.$$

Then for $t \geq t_0$

$$\frac{|f(tx_t) - f(tx)|}{f(te)} \leq \frac{|f(tx_t) - f(tx)|}{cf(tx)} = \frac{1}{c}\left|\frac{f(tx_t)}{f(tx)} - 1\right|.$$

From the last relation it follows that (1.4.11) holds true for an arbitrary vector $e \in S$. The theorem is proved. □

With the use of Theorem 1.4.5, we arrive at the following.

COROLLARY 1.4.2. *The assertions below are equivalent.*

(a) *A function f is weakly oscillating.*

(b) (1.4.11) *holds true for any vector $e \in S$.*

(c) (1.4.11) *holds true for any vector $e \in S$, $|e| = 1$.*

(d) (1.4.11) *holds true for some vector $e \in S$, and*

$$\liminf_{t\to\infty} \frac{f(tx)}{f(te)} > 0$$

for any $x \in S$, $|x| = 1$.

We recall that a cone Γ is said to be solid if $\operatorname{int}\Gamma \neq \varnothing$.

COROLLARY 1.4.3. *Let f be feebly oscillating at infinity in Γ ((1.4.11) holds). Then there exists a solid cone R with apex at zero such that $e \in \operatorname{int} R$ and f is weakly oscillating at infinity in $\Gamma \cap R$.*

PROOF. Let f obey (1.4.11). Then by (1.4.13) there exist $\delta > 0$ and $t_0 > 0$ such that

$$|f(tx)/f(te) - 1| \le 1/2 \tag{1.4.21}$$

for all $t \ge t_0$, $x \in S$, $|x - e| \le \delta$. We set

$$R = \{y \colon y = \lambda x,\ x \in B,\ \lambda > 0\},$$

where

$$B = \{x \colon x \in \mathbf{R}^n,\ |x - e| \le \delta\}.$$

It is clear that R is a solid cone, $e \in \operatorname{int} R$. We choose an arbitrary vector $y \in S \cap R$, $y = \lambda x$ for some $\lambda > 0$ and $x \in B$. By virtue of Corollary 1.4.1, there exists $t_1 > 0$ such that

$$f(t\lambda e)/f(te) > c > 0 \tag{1.4.22}$$

for all $t \ge t_1$ with some constant c. From (1.4.21) and (1.4.22) it follows that

$$\frac{f(ty)}{f(te)} = \frac{f(t\lambda x)}{f(t\lambda e)} \frac{f(t\lambda e)}{f(te)} \ge \frac{c}{2}$$

for $t \ge \max(t_0/\lambda, t_1)$. In order to complete the proof of Corollary 1.4.3, it remains to make use of Corollary 1.4.2. □

It is clear that in the one-dimensional case the feebly oscillating functions are weakly oscillating. The example below demonstrates that this is not the case for $n > 1$. Let

$$\Gamma = \{x = (x_1, x_2),\ x_1 \ge 0,\ x_2 > 0\},$$

$$f(x) = \begin{cases} x_1, & x_1 > 0, \\ \ln x_2, & x_1 = 0, \end{cases}$$

where $x = (x_1, x_2) \in \Gamma$. It is not difficult to see that f obeys (1.4.11) with $e = (1, 1)$, that is, f feebly oscillates. But for $x = (0, 1)$

$$f(tx)/f(te) = \ln t/t \to 0, \qquad t \to \infty,$$

and therefore, f is not weakly oscillating.

The theorem below extends the well-known integral representation theorem on regularly varying functions in (Seneta, 1976).

THEOREM 1.4.6. *If a cone Γ is closed, then*

(1) *a function f is feebly oscillating at infinity in Γ if and only if it admits the representation*

$$f(x) = h(x) \exp\left(\int_b^{|x|} \frac{\varepsilon(t)}{t} dt\right), \tag{1.4.23}$$

where $x \in \Gamma$, $|x| \ge b > 0$, and $h(x)$ is feebly oscillating at infinity in Γ, $\varepsilon(t)$ is measurable, and

$$\sup_{|x| \ge b} h(x) < \infty, \qquad \sup_{t \ge b} |\varepsilon(t)| < \infty.$$

(2) *A function* $f(x)$ *is weakly oscillating at infinity in* Γ *if and only if it admits the representation*

$$f(x) = \exp\left(\eta(x) + \int_b^{|x|} \frac{\varepsilon(t)}{t}dt\right), \tag{1.4.24}$$

where $|x| \geq b > 0$, $x \in \Gamma$, *and* $\varepsilon(t)$ *is measurable,*

$$\sup_{|x|\geq b} |\eta(x)| < \infty, \qquad \sup_{t\geq b} |\varepsilon(t)| < \infty,$$

and the function $\exp(\eta(x))$ *weakly oscillates at infinity in* Γ.

PROOF. For $n = 1$ the theorem follows from the corresponding assertion concerning the *RO-varying functions* (Seneta, 1976, Theorem A1). We recall that a positive and measurable for $x \geq a \geq 0$ function $f(x)$ of one variable is said to be *RO*-varying at infinity if for any $\lambda > 0$

$$0 < \liminf_{x\to\infty} \frac{f(\lambda x)}{f(x)} \leq \limsup_{x\to\infty} \frac{f(\lambda x)}{f(x)} < \infty.$$

Let $n > 1$ and the function f be weakly oscillating at infinity in Γ. We choose some vector $e \in S$ and represent f as

$$f(x) = \frac{f(x)}{f(|x|e)} f(|x|e).$$

In view of closeness of Γ, the set $B = \{x : x \in \Gamma,\ |x| = 1\}$ is a compact. From this fact and Theorem 1.4.5 it follows that

$$0 < \liminf_{|x|\to\infty} \frac{f(x)}{f(|x|e)} \leq \limsup_{|x|\to\infty} \frac{f(x)}{f(|x|e)} < \infty,$$

and the function $f(x)/f(|x|e)$ is weakly oscillating as the ratio of two weakly oscillating functions. It remains, because, as we have seen, the theorem is true for $n = 1$, to represent $f(|x|e)$ in the form (1.4.24). Part 2 of the theorem is proved; part 1 is validated similarly. □

COROLLARY 1.4.4. *Let a cone* Γ *be closed and a function* $f(x)$ *be weakly oscillating at infinity in* Γ. *Then there exist real* α, β, $b > 0$, $c > 0$, $c_0 > 0$ *such that*

$$c_0|x|^\alpha \leq \frac{f(tx)}{f(te)} \leq c|x|^\beta, \qquad \forall x \in \Gamma : |x| \geq 1, \tag{1.4.25}$$

$$c_0|x|^\beta \leq \frac{f(tx)}{f(te)} \leq c|x|^\alpha, \qquad \forall x \in \Gamma : b/t \leq |x| \leq 1 \tag{1.4.26}$$

for all $t \geq b$ *and* $e \in \Gamma$, $|e| = 1$.

PROOF. In accordance with representation (1.4.24) for $t \geq b$, $|e| = 1$, $|x| \geq 1$,

$$\frac{f(tx)}{f(te)} = \exp\left(\eta(tx) - \eta(te) + \int_t^{t|x|} \frac{\varepsilon(u)}{u}du\right)$$

$$\leq c\exp\left(\beta\int_t^{t|x|} \frac{1}{u}du\right) = c\exp(\beta\ln|x|) = c|x|^\beta$$

with some constants c and β. If $b/t \le |x| \le 1$, then for $t \ge b$ and $|e| = 1$

$$\frac{f(tx)}{f(te)} \le c \exp\left(-\int_{t|x|}^{t} \frac{\varepsilon(u)}{u} du\right) \le c \exp\left(-\alpha \int_{t|x|}^{t} \frac{du}{u}\right) = c|x|^{\alpha}$$

with some real α. The left-hand sides of inequalities (1.4.25) and (1.4.26) are proved similarly. □

REMARK 1.4.4. By virtue of Corollary 1.4.4, if a cone Γ is closed, then for any weakly oscillating at infinity in Γ function $f(x)$ the indices

$$\begin{aligned}\operatorname{ind}_- f &= \sup\{\alpha : \alpha \in \mathbf{R},\ \exists b, c > 0 \mid \forall t \ge b,\ \forall x \in S,\ b/t \le |x| \le 1, \\ &\qquad f(tx)/f(te) \le c|x|^{\alpha}\},\\ \operatorname{ind}_+ f &= \inf\{\beta : \beta \in \mathbf{R},\ \exists b, c > 0 \mid \forall t \ge b,\ \forall x \in S,\ |x| \ge 1,\ f(tx)/f(te) \le c|x|^{\beta}\}\end{aligned}$$

are finite and do not depend on the vector $e \in S$.

To close this section, we speak about one more class of functions which will find use in Tauberian theorems.

DEFINITION 1.4.5. For all $t \ge t_0 \ge 0$, let some positive function $r(t)$ be defined. A complex-valued function $f(x)$ defined for $x \in \Gamma$, $|x| \ge a \ge 0$, is said to be *r-slowly varying* (at infinity in Γ) if for all $x \in S$

$$f(tx_t) - f(tx) = o(r(t)), \qquad t \to \infty, \tag{1.4.27}$$

as $x_t \to x$, $x_t \in S$.

We highlight two particular cases of (1.4.27). First, for $r(t) \equiv 1$ and $\Gamma = \{t : t \ge 0\}$ we arrive at a well-known notion of a *slowly oscillating function* (see, e.g., (Postnikov, 1988; Postnikov, 1980)). Second, for $f(x) > 0$ and $r(t) = f(te)$ with some $e \in S$ we find ourselves in the class of feebly oscillating functions considered above (see Definition 1.4.3). The following assertion is true.

THEOREM 1.4.7. *Let a function $f(x)$ be r-slowly oscillating at infinity in Γ. Then*

$$\limsup_{t\to\infty} \sup_{x,y \in K} \frac{|f(ty) - f(tx)|}{r(t)} < \infty \tag{1.4.28}$$

for any compact $K \subseteq S$, and the limit

$$\lim_{\substack{\delta\to 0\\ t\to\infty}} \sup_{\substack{x,y\in K\\ |x-y|\le\delta}} \frac{|f(tx) - f(ty)|}{r(t)} = 0 \tag{1.4.29}$$

exists.

Theorem 1.4.7 is proved in the same way as Theorem 1.4.5.

In the one-dimensional case, close classes of functions were studied in (Bingham, Goldie, 1982a; Bingham, Goldie, 1982b; de Haan, 1970; Geluk, de Haan, 1987; de Haan, Stadtmüller, 1985). In the book (Geluk, de Haan, 1987), a measurable function $f\colon \mathbf{R}_+ \to \mathbf{R}$ is called *asymptotically balanced* if there exists a function $a\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\varphi(x) = \limsup_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} < \infty, \qquad \forall x > 1,$$
$$\psi(x) = \liminf_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} > -\infty, \qquad \forall x > 0,$$

and there exists $x_0 > 1$ such that

$$\psi(x) = \liminf_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} > 0, \qquad \forall x \geq x_0.$$

Even more general classes of functions were considered in (Bingham, Goldie, 1982a; Bingham, Goldie, 1982b). It was assumed there that the above function $\varphi(x)$ is merely finite on a set of x inside $[1, \infty)$ of positive Lebesgue measure.

Of much probabilistic application are the so-called *π-varying functions* introduced in (de Haan, 1970) which comprise a special case of asymptotically balanced ones such that $\varphi(x) = \psi(x) = \ln x \ \forall x > 0$. The π-varying functions of two variables are considered in (Omey, 1989). In the context of renewal theory, weakly oscillating functions of one variable are recently studied in (Buldygin *et al.*, 2002).

1.5. A multidimensional Tauberian comparison theorem

Let Γ be a closed convex acute solid cone in $\mathbf{R}^n$ with apex at zero (see the beginning of Section 1.1), and let Γ^* be the dual to Γ cone:

$$\Gamma^* = \{y\colon y \in \mathbf{R}^n, (y, x) \geq 0 \ \forall x \in \Gamma\}.$$

We preserve the notation of Section 1.3: $S = \Gamma \setminus \{0\}$, $G = \operatorname{int}\Gamma$, $C = \operatorname{int}\Gamma^*$; the relations $x \overset{\Gamma}{\leq} y$ and $x \overset{\Gamma}{<} y$ mean, respectively, that $x, y, y - x \in \Gamma$ and that $x \in \Gamma$, $y, y - x \in G$; the Laplace transform of a function f and the Laplace–Stieltjes transform of a measure F on Γ are denoted, respectively, by $\widehat{f}(\lambda)$ and $\widetilde{F}(\lambda)$:

$$\widehat{f}(\lambda) = \int_\Gamma e^{-(\lambda,x)} f(x)\, dx, \qquad \widetilde{F}(\lambda) = \int_\Gamma e^{-(\lambda,x)} F(dx)$$

(under the assumption that they exist for $\lambda \overset{C}{>} a$ with some $a \in \Gamma^*$). Since the Laplace–Stieltjes transform reduces to the Laplace transform as follows:

$$\widetilde{F}(\lambda) = \widehat{f}(\lambda)/\widehat{1}(\lambda), \qquad \lambda \in C, \tag{1.5.1}$$

where $f(x) = F\{y\colon y \overset{\Gamma}{\leq} x\}$, $\widehat{1}(\lambda)$ is the Laplace transform of the unit function, we will deal with Tauberian theorems on the Laplace transforms only.

Before formulating and proving multidimensional Tauberian theorems, we prove the following two continuity theorems.

THEOREM 1.5.1. *Let complex-valued functions $f_m(x)$, $f(x)$ be defined and measurable in Γ, $|f_m(x)| \leq \varphi(x)$ for all $x \in \Gamma$ and $m \in \mathbf{N}$,*

$$\widehat{\varphi}(a) < \infty \tag{1.5.2}$$

for some $a \in C$, and let $f_m(x) \to f(x)$ as $m \to \infty$ almost everywhere in Γ (with respect to the Lebesgue measure). Then for all $\lambda \overset{C}{>} a$ and $m \in \mathbf{N}$ there exist the Laplace transforms $\widehat{f}_m(\lambda)$, $\widehat{f}(\lambda)$, and

$$\widehat{f}_m(\lambda) \to \widehat{f}(\lambda), \qquad m \to \infty.$$

PROOF. It is clear that f_m, f are Lebesgue-integrable on an arbitrary set A of the form $A = \{x\colon x \in \Gamma,\ |x| \leq t\}$, and there exist the Laplace transforms $\widehat{f}_m(\lambda)$ and $\widehat{f}(\lambda)$ for $\lambda \overset{C}{>} a$. Further, by virtue of Lebesgue's theorem,

$$\int_A e^{-(\lambda,x)} f_m(x)\,dx \to \int_A e^{-(\lambda,x)} f(x)\,dx, \qquad m \to \infty, \tag{1.5.3}$$

for any $\lambda \in C$. We fix some $\lambda \overset{C}{>} a$ and $\varepsilon > 0$. For these λ and ε we choose t in such a way that

$$\int_B e^{-(\lambda,x)} \varphi(x)\,dx \leq \varepsilon/3,$$

where $B = \Gamma \setminus A$. Then

$$\left|\int_B f_m(x) e^{-(\lambda,x)} dx\right| \leq \int_B \varphi(x) e^{-(\lambda,x)} dx \leq \varepsilon/3,$$
$$\left|\int_B f(x) e^{-(\lambda,x)} dx\right| \leq \int_B \varphi(x) e^{-(\lambda,x)} dx \leq \varepsilon/3.$$

By (1.5.3), we choose $k \in \mathbf{N}$ so that for $m \geq k$

$$\left|\int_A e^{-(\lambda,x)} (f_m(x) - f(x))\,dx\right| \leq \varepsilon/3.$$

From the above inequalities for all $m \geq k$ we obtain the inequality

$$|\widehat{f}_m(\lambda) - \widehat{f}(\lambda)| \leq \varepsilon.$$

The theorem is proved. □

THEOREM 1.5.2. *Let a sequence of complex-valued functions $f_m(x)$ defined in Γ be asymptotically continuous in D ($D = G$ or $D = S$) (see Definition* 1.4.1*), and $|f_m(x)| \leq \varphi(x)$ for all $m \in \mathbf{N}$ and $x \in D$, while the function $\varphi(x)$ obeys* (1.5.2) *with some $a \in \Gamma^*$; further, for all $\lambda \overset{C}{>} a$ let, as $m \to \infty$,*

$$\widehat{f}_m(\lambda) \to \omega(\lambda), \qquad |\omega(\lambda)| < \infty.$$

Then there exists a function $f(x)$ defined in D with a finite Laplace transform, $|\widehat{f}(\lambda)| < \infty$, for $\lambda \overset{C}{>} a$, such that for any compact $K \subseteq D$

$$f_m(x) \to f(x), \qquad m \to \infty,$$

uniformly in $x \in K$, and $\omega(\lambda) = \widehat{f}(\lambda)$ for all $\lambda \overset{C}{>} a$.

PROOF. We take arbitrary $x \in D$ and $\lambda \overset{C}{>} a$. Then for sufficiently large $m > k$

$$\begin{aligned} |f_m(x)| &\le \left| f_m(x) - |A|^{-1} \int_A e^{-(\lambda,y)} f_m(y)\,dy \right| + |A|^{-1} \left| \int_A e^{-(\lambda,y)} f_m(y)\,dy \right| \\ &\le |A|^{-1} \int_A e^{-(\lambda,y)} |f_m(x) - f_m(y)|\,dy + |A|^{-1} |\widehat{f}_m|(\lambda) \\ &\le \sup_{y \in A} |f_m(x) - f_m(y)| + |A|^{-1} \widehat{\varphi}(\lambda) < c < \infty, \end{aligned}$$

where $A = \{y \colon y \in D,\ |y - x| \le \varepsilon\}$, $|A|$ is the Lebesgue measure of A, and $\varepsilon > 0$ is appropriately chosen. The sequence $\{f_m(x),\ m \in \mathbf{N}\}$ is thus pre-compact in D in the pointwise convergence topology (Theorem 1.4.1). For some unbounded set $L \subseteq \mathbf{N}$ and all $x \in D$ let, as $m \to \infty$, $m \in L$,

$$f_m(x) \to f(x). \tag{1.5.4}$$

Then by virtue of Theorem 1.5.1 for $m \in L$, as $m \to \infty$,

$$\widehat{f}_m(\lambda) \to \widehat{f}(\lambda)$$

for all $\lambda \overset{C}{>} a$. Therefore, $\widehat{f}(\lambda) = \omega(\lambda)$. Since f is continuous (Theorem 1.4.2), it is uniquely determined by its Laplace transform $\omega(\lambda)$. Since the limit function in (1.5.4) does not depend on the choice of the subsequence $L \subseteq \mathbf{N}$, (1.5.4) holds as $m \to \infty$, $m \in \mathbf{N}$. From Theorem 1.4.2 it also follows that (1.5.4) holds uniformly in $x \in K$ for any compact $K \subseteq D$. □

First we prove the following Tauberian theorem of Littlewood type (Littlewood, 1910).

THEOREM 1.5.3. *Let a function $r(t)$ be regularly varying at infinity with index $\gamma > -n$ (see (Seneta, 1976)), a function $f(x)$ be r-slowly varying at infinity in Γ (see Definition* 1.4.5*), and for all $\lambda \in C$ let $|\widehat{f}(\lambda)| < \infty$. Then the following assertions are true.*

(1) *If*

$$\frac{\widehat{f}(\lambda/t)}{t^n r(t)} \to \psi(\lambda), \qquad |\psi(\lambda)| < \infty \tag{1.5.5}$$

for all $\lambda \in C$ as $t \to \infty$, then

$$f(tx)/r(t) \to \varphi(x), \qquad |\varphi(x)| < \infty \tag{1.5.6}$$

for all $x \in S$ as $t \to \infty$; further, for any $\lambda \in C$ there exists $\widehat{\varphi}(\lambda)$, and

$$\widehat{\varphi}(\lambda) = \psi(\lambda). \tag{1.5.7}$$

(2) *If* (1.5.6) *holds, then* (1.5.5) *and* (1.5.7) *also hold with some function* $\psi(\lambda)$.

(3) *Under the hypotheses of assertions* 1 *and* 2*, the function* $\varphi(x)$ *is continuous and homogeneous in* S *with homogeneity degree* γ *(that is,* $\varphi(tx) = t^\gamma \varphi(x)$ *for* $t > 0$ *and* $x \in S$*), and relation* (1.5.6) *holds uniformly in* $x \in K$ *for any compact* $K \subseteq S$.

PROOF. We fix some $\varepsilon > 0$ so that $\gamma - \varepsilon > -n$. From the integral representation theorem for regularly varying functions it follows (Vladimirov, Zavyalov, 1981) that there exists $t_0 > 0$ such that

$$h(u) \le r(ut)/r(t) \le g(u) \tag{1.5.8}$$

for all $t \ge t_0$ and $u \ge t_0/t$, where

$$g(u) = \begin{cases} u^{\gamma+\varepsilon}, & u > 1+\varepsilon, \\ u^{\gamma-\varepsilon}, & u < 1-\varepsilon, \end{cases} \qquad h(u) = \begin{cases} u^{\gamma-\varepsilon}, & u > 1+\varepsilon, \\ u^{\gamma+\varepsilon}, & u < 1-\varepsilon, \end{cases}$$

and the functions $g(u)$, $h(u)$ take some constant values on the interval $[1-\varepsilon, 1+\varepsilon]$.

First we prove assertion 2. By virtue of relations (1.4.28), (1.5.6), and (1.5.8), there exist $t_1 > t_0$, $c < \infty$ such that

$$\begin{aligned} \frac{|f(tx)|}{r(t)} &= \frac{|f(tx)|}{r(t|x|)} \frac{r(t|x|)}{r(t)} \le \frac{|f(\tau x/|x|)|}{r(\tau)} g(|x|) \\ &\le \frac{f(\tau e)}{r(\tau)} g(|x|) + \frac{|f(\tau x/|x|) - f(\tau e)|}{r(\tau)} g(|x|) \le c g(|x|) \end{aligned} \tag{1.5.9}$$

for all $t \ge t_1$, $x \in \Gamma$, $|x| \ge t_1/t$, where $\tau = t|x|$, and some $e \in S$. For $t \ge t_1$ and $x \in \Gamma$, we set

$$g_t(x) = \begin{cases} f(tx)/r(t), & |x| \ge t_1/t, \\ 0, & |x| < t_1/t. \end{cases}$$

We observe that the family of functions $\{g_t(x),\ t \ge t_1\}$ is asymptotically continuous in S as $t \to \infty$ (see Definition 1.4.2 and Remark 1.4.1), and in view of (1.5.6) and (1.5.9), for all $x \in S$ as $t \to \infty$

$$g_t(x) \to \varphi(x),$$

and $|g_t(x)| \le c g(|x|)$ for $t \ge t_1$. There exists, obviously, the Laplace transform of the function $g(|x|)$ in C. Hence, with the use of Theorem 1.5.1, for all $\lambda \in C$ we obtain

$$\widehat{g}_t(\lambda) \to \widehat{\varphi}(\lambda), \qquad t \to \infty. \tag{1.5.10}$$

But as $t \to \infty$

$$\begin{aligned} \widehat{g}_t(\lambda) &= \int_{|x| \ge t_1/t} \frac{f(tx)}{r(t)} e^{-(\lambda,x)} dx = \int_{|y| \ge t_1} e^{-(\lambda/t,y)} \frac{f(y)}{r(t)} t^{-n} dy \\ &= \frac{\widehat{f}(\lambda/t)}{t^n r(t)} - \int_{|y| \le t_1} \frac{f(y)}{t^n r(t)} e^{-(\lambda/t,y)} dy = \frac{\widehat{f}(\lambda/t)}{t^n r(t)} + o(1), \end{aligned} \tag{1.5.11}$$

because $t^n r(t) \to \infty$ as $t \to \infty$ and

$$\left| \int_{|y| \le t_1} e^{-(\lambda/t, y)} f(y)\, dy \right| \le \int_{|y| \le t_1} |f(y)|\, dy = \text{const.}$$

The desired result now follows from relations (1.5.10) and (1.5.11). Assertion 2 of the theorem is proved.

Now let us turn to the proof of assertion 1. As for assertion 2, we see that

$$I(t_2) \equiv \int_{|x| \ge t_2/t} \frac{f(tx)}{r(t)} e^{-(\lambda,x)} dx \to \psi(\lambda) \tag{1.5.12}$$

for any $t_2 > 0$ as $t \to \infty$. We fix some vector $e \in S$. We assume that there is an unbounded set $T \subseteq (0, \infty)$ such that

$$\frac{|f(te)|}{r(t)} \to +\infty, \quad t \to \infty, \quad t \in T.$$

Without loss of generality we assume that there is an unbounded set $T_1 \subseteq T$ such that

$$\frac{|\Re f(te)|}{r(t)} \to +\infty, \quad t \to \infty, \quad t \in T_1;$$

otherwise we turn to the function if. We also assume that there is an unbounded set $T_2 \subseteq T_1$ such that

$$\Re f(te)/r(t) \to +\infty, \quad t \to \infty, \quad t \in T_2; \tag{1.5.13}$$

otherwise we turn to the function $-f$. Furthermore, in view of (1.5.8), (1.5.12), and (1.4.28) there exists $t_3 > t_2$ such that for any $\tau \ge t_3$

$$\begin{aligned} &\int_{|x| \ge \tau/t} \frac{f(t|x|e)}{r(t)} e^{-(\lambda,x)} dx \\ &\qquad = \int_{|x| \ge \tau/t} \frac{r(t|x|)}{r(t)} \frac{(f(tx) - f(t|x|e))}{r(t|x|)} e^{-(\lambda,x)} dx + I(\tau) = O(1) \end{aligned} \tag{1.5.14}$$

as $t \to \infty$. By (1.5.13), to each $M > 0$ we put in correspondence some $\tau = \tau(M)$ in such a way that for $t \ge \tau$, $t \in T_2$ the formula

$$\Re \frac{f(te)}{r(t)} \ge M \tag{1.5.15}$$

is true. Then by virtue of (1.5.8) and (1.5.15)

$$\begin{aligned} \Re \int_{|x| \ge \tau/t} \frac{f(t|x|e)}{r(t)} e^{-(\lambda,x)} dx &= \int_{|x| \ge \tau/t} \Re \frac{f(t|x|e)}{r(t|x|)} \frac{r(t|x|)}{r(t)} e^{-(\lambda,x)} dx \\ &\ge M \int_{|x| \ge \tau/t} e^{-(\lambda,x)} h(|x|)\, dx. \end{aligned}$$

Therefore,

$$\liminf_{t\to\infty,\ t\in T_2} \Re \int_{|x|\geq\tau/t} \frac{f(t|x|e)}{r(t)} e^{-(\lambda,x)} dx \geq M \int_{\Gamma} e^{-(\lambda,x)} h(|x|)\, dx$$

We set $t_4 = \tau(0)$. The inequality is just weakened:

$$\liminf_{t\to\infty,\ t\in T_2} \Re \int_{|x|\geq t_4/t} \frac{f(t|x|e)}{r(t)} e^{-(\lambda,x)} dx \geq M \int_{\Gamma} e^{-(\lambda,x)} h(|x|)\, dx.$$

Since the left-hand side of the last inequality does not depend on M, we see that

$$\Re \int_{|x|\geq t_4/t} \frac{f(t|x|e)}{r(t)} e^{-(\lambda,x)} dx \to +\infty, \quad t\to\infty, \quad t\in T_2,$$

which contradicts (1.5.14). Hence,

$$\limsup_{t\to\infty} \frac{|f(te)|}{r(t)} < \infty.$$

Therefore, as for assertion 2, we conclude that (1.5.9) is valid. The rest follows from Theorem 1.5.2. Assertion 3, in view of the abovesaid, is an immediate corollary to Theorem 1.5.2. □

For two cones Γ_1 and Γ_2 we write $\Gamma_1 \prec \Gamma_2$ if the closure of the set $\{x\colon x\in\Gamma_1,\ |x|=1\}$ is contained in $\operatorname{int}\Gamma_2$. The main result of this section is the following multidimensional Tauberian comparison theorem.

THEOREM 1.5.4. *For some non-negative functions $f(x)$ and $g(x)$ defined in Γ, let there exist their Laplace transforms $\widehat{f}(\lambda)$ and $\widehat{g}(\lambda)$ for $\lambda\in C$, let the function $f(x)$ be weakly oscillating at infinity in Γ (see Definition* 1.4.4*), let $g(x) = r(x)h(x)$, where $r(x)$ is monotone inside G (see the beginning of Section* 1.2*), and let the function $h(x)$ be weakly oscillating in G and* $\operatorname{ind}_- f > -n$ *(see Remark* 1.4.4*). If*

$$\widehat{g}(\lambda t)/\widehat{f}(\lambda t) \to 1, \qquad t\to 0_+, \tag{1.5.16}$$

for some solid cone $C_0 \prec C$ and all $\lambda\in C_0$, then

$$g(x)/f(x) \to 1, \quad |x|\to\infty, \quad x\in\Gamma_0, \tag{1.5.17}$$

for any cone $\Gamma_0 \prec \Gamma$.

This Tauberian theorem extends those given in (Vladimirov, 1978; Stadtmüller, Trautner, 1979; Stadtmüller, Trautner, 1981; Stadtmüller, 1983).

PROOF. We assume the contrary: let there exist a sequence $x_m \in \Gamma_0$ such that $|x_m|\to\infty$ as $m\to\infty$, and a constant $c\neq 1$, $0\leq c\leq\infty$, such that

$$g(x_m)/f(x_m) \to c, \qquad m\to\infty. \tag{1.5.18}$$

We set $x_m = t_m e_m$, where $t_m = |x_m|$, $e_m = x_m/|x_m|$. Without loss of generality, we assume that $e_m \to e \in B = \{x : x \in \Gamma,\ |x| = 1\}$ as $m \to \infty$. We observe that $e \in G$, because $\Gamma_0 \prec \Gamma$. Further, for $\lambda \in C_0$

$$\widehat{f}(\lambda/t_m) = \int_\Gamma e^{-(\lambda, x/t_m)} f(x)\, dx = t_m^n f(x_m) \int_\Gamma \frac{f(t_m x)}{f(t_m e_m)} e^{-(\lambda,x)} dx. \tag{1.5.19}$$

We observe that the sequence of functions

$$a_m(x) = f(t_m x)/f(x_m)$$

is asymptotically continuous in S (in the sense of Definition 1.4.1). It is easily seen indeed that, by virtue of Theorems 1.4.5 and 1.4.4

$$a_m(x) - a_m(y) = \frac{f(t_m x) - f(t_m y)}{f(t_m e_m)} = (1 + o(1)) \frac{f(t_m x) - f(t_m y)}{f(t_m e)} = o(1) \tag{1.5.20}$$

as $m \to \infty$, $y \to x$, $x, y \in S$. Next, by virtue of Corollary 1.4.4, there are α, β, $l \geq 0$, $c_1 > 0$, $-n < \alpha \leq \beta < \infty$, such that for all $t \geq l$ and $m \in \mathbf{N}$

$$f(tx)/f(te_m) \leq \varphi(x), \tag{1.5.21}$$

where

$$\varphi(x) = \begin{cases} c_1 |x|^\beta, & |x| \geq 1, \\ c_1 |x|^\alpha, & l/t \leq |x| \leq 1. \end{cases}$$

We set

$$b_m(x) = \begin{cases} a_m(x), & |x| \geq l/t_m, \\ 0, & |x| < l/t_m. \end{cases}$$

In view of (1.5.20), the sequence of functions $\{b_m(x),\ m \in \mathbf{N}\}$ is asymptotically continuous in S, and for all $m \in \mathbf{N}$, by (1.5.21),

$$b_m(x) \leq \varphi(x), \tag{1.5.22}$$

and since $\alpha > -n$, we see that $\widehat{\varphi}(\lambda) < \infty$ for all $\lambda \in C$. Taking account for Theorems 1.4.1 and 1.4.2, without loss of generality we assume that

$$b_m(x) \to a(x) \tag{1.5.23}$$

as $m \to \infty$ for all $x \in S$ and some continuous function $a(x)$. In this case, by virtue of Theorem 1.5.1, taking into account relations (1.5.22) and (1.5.23), we find that for all $\lambda \in C$ as $m \to \infty$

$$\widehat{b}_m(\lambda) \to \widehat{a}(\lambda) < \infty. \tag{1.5.24}$$

Further, in view of (1.5.24),

$$\begin{aligned} \int_\Gamma \frac{f(t_m x)}{f(t_m e_m)} e^{-(\lambda,x)} dx &= \widehat{b}_m(\lambda) + \int_{|x| \leq l} \frac{f(x)}{t_m^n f(t_m e_m)} e^{-(\lambda/t_m, x)} dx \\ &= \widehat{a}(\lambda) + o(1) \end{aligned} \tag{1.5.25}$$

as $m \to \infty$, because, as $|x| \to \infty$,

$$|x|^n f(x) \to +\infty$$

by virtue of Corollary 1.4.4 and the fact that $\mathrm{ind}_- f > -n$. Then

$$\widehat{g}(\lambda/t_m) = t_m^n f(x_m) \int_\Gamma \frac{g(t_m x)}{f(x_m)} e^{-(\lambda,x)} dx. \tag{1.5.26}$$

Therefore, from (1.5.16), (1.5.19), (1.5.25), and (1.5.26) it follows that as $m \to \infty$

$$\int_\Gamma \frac{g(t_m x)}{f(t_m e_m)} e^{-(\lambda,x)} dx \to \widehat{a}(\lambda) \tag{1.5.27}$$

for any $\lambda \in C_0$. We set

$$v_m(x) = \frac{h(t_m x)}{h(t_m e)}, \qquad u_m(x) = \frac{r(t_m x) h(t_m e)}{f(x_m)}.$$

It is clear that the functions $u_m(x)$ are monotone inside G and the sequence of functions $\{v_m(x),\ m \in \mathbf{N}\}$ is asymptotically continuous in G. By virtue of (1.5.27) and Theorem 1.3.2, as $m \to \infty$

$$\int_A u_m(y) v_m(y)\, dy \to \int_A a(y)\, dy, \tag{1.5.28}$$

where

$$A = \{y \colon y \in \Gamma,\ y \overset{\Gamma}{\geq} x,\ |y - x| \leq \varepsilon\}.$$

We observe that

$$u_m(x) \leq c_2 \tag{1.5.29}$$

for some $c_2 < \infty$ and all $m \in \mathbf{N}$. In view of (1.5.28), there indeed exists $c_3 < \infty$ such that for all $m \in \mathbf{N}$

$$\int_A u_m(y) v_m(y)\, dy \leq c_3,$$

hence by monotonicity of $u_m(y)$ (for the sake of definiteness, let $r(y)$ do not decrease)

$$u_m(x) \int_A v_m(y)\, dy \leq c_3,$$

and therefore,

$$u_m(x) |A| \inf_{y \in A} v_m(y) \leq c_3.$$

But by virtue of Theorem 1.4.5 there is a constant $c_4 > 0$ such that for some $m_1 \in \mathbf{N}$

$$\inf_{y \in A} v_m(y) = \inf_{y \in A} \frac{h(t_m y)}{h(t_m e)} > c_4$$

for all $m > m_1$. The last two inequalities yield (1.5.29). Let

$$U_m(dx) = \frac{g(t_m x)}{f(x_m)} dx, \qquad U(dx) = a(x)\,dx.$$

Then by (1.5.29)

$$\begin{aligned}
\frac{g(t_m x)}{f(x_m)} &= \frac{1}{|A|}\int_A \frac{g(t_m x)}{f(x_m)} dy = \frac{U_m(A)}{|A|} - \frac{1}{|A|}\int_A \left(\frac{g(t_m x) - g(t_m y)}{f(x_m)}\right) dy \\
&= \frac{1}{|A|}\left(U_m(A) - \int_A (u_m(y)v_m(y) - u_m(x)v_m(x))\,dy\right) \\
&\le \frac{1}{|A|}\left(U_m(A) - \int_A (u_m(x)v_m(y) - u_m(x)v_m(x))\,dy\right) \\
&= \frac{1}{|A|}\left(U_m(A) - u_m(x)\int_A (v_m(y) - v_m(x))\,dy\right) \\
&\le \frac{U_m(A)}{|A|} + c_2 \sup_{y\in A} |v_m(y) - v_m(x)|. \qquad (1.5.30)
\end{aligned}$$

Therefore, by (1.5.28) and (1.5.30),

$$\limsup_{m\to\infty} \frac{g(t_m x)}{f(x_m)} \le \frac{U(A)}{|A|} + c_2 \limsup_{m\to\infty} \sup_{y\in A} |v_m(y) - v_m(x)|.$$

Passing to the limit as $\varepsilon \to 0$ in the last inequality and recalling the asymptotic continuity of the sequence of functions $\{v_m(y),\ m \in \mathbf{N}\}$, we obtain

$$\limsup_{m\to\infty} \frac{g(t_m x)}{f(x_m)} \le a(x).$$

Using lower bounds similar to (1.5.30) (provided that the set A is appropriately changed), we arrive at

$$g(t_m x)/f(t_m e_m) \to a(x), \qquad m \to \infty. \qquad (1.5.31)$$

By (1.5.23) and the definition of the functions $b_m(x)$,

$$1 = \lim_{m\to\infty} \frac{f(t_m e)}{f(t_m e_m)} = a(e). \qquad (1.5.32)$$

For $x = e$, from (1.5.31) with account for (1.5.32) we obtain

$$\frac{g(t_m e)}{f(t_m e_m)} \to 1, \qquad m \to \infty.$$

But if $\nu > 1$, by virtue of (1.5.29) with $x = \nu e$, for m large enough

$$\begin{aligned}
\frac{g(t_m e_m)}{f(t_m e_m)} &= u_m(e_m)v_m(e_m) \le u_m(\nu e)v_m(e_m) \\
&= u_m(\nu e)v_m(\nu e) + u_m(\nu e)(v_m(e_m) - v_m(\nu e) \\
&\le u_m(\nu e)v_m(\nu e) + c_2|v_m(e_m) - v_m(\nu e)|.
\end{aligned}$$

Hence

$$\limsup_{m\to\infty} \frac{g(t_m e_m)}{f(t_m e_m)} \le f(\nu e) + c_2 \limsup_{m\to\infty} |v_m(e_m) - v_m(\nu e)|;$$

passing to the limit as $\nu \downarrow 1$ and recalling the asymptotic continuity of $\{v_m(x),\ m \in \mathbf{N}\}$, we obtain

$$\limsup_{m\to\infty} \frac{g(t_m e_m)}{f(t_m e_m)} \le 1.$$

Using similar lower bounds, we see that, as $m \to \infty$,

$$g(t_m e_m)/f(t_m e_m) = g(x_m)/f(x_m) \to 1,$$

which contradicts (1.5.18). The theorem is proved. □

The following Abelian theorem is true.

THEOREM 1.5.5. *For non-negative functions $f(x)$ and $g(x)$ defined in Γ, let there exist their Laplace transforms $\widehat{f}(\lambda)$ and $\widehat{g}(\lambda)$ for $\lambda \in C$,*

$$f(x)/g(x) \to c$$

for some $c \ge 0$ as $x \to \infty$, $x \in \Gamma$, and $\widehat{g}(\lambda) \to \infty$ as $\lambda \to 0$, $\lambda \in C$. Then

$$\widehat{f}(\lambda)/\widehat{g}(\lambda) \to c$$

as $\lambda \to 0$, $\lambda \in C$.

PROOF. We assume that $c > 0$. It is obvious that we may set $c = 1$. For $|x| \ge a$, let

$$f(x) = g(x)(1 + \delta(x)),$$

where $|\delta(x)| \le \varepsilon$ for $|x| \ge a$. Then for $\lambda \in C$

$$\begin{aligned} |\widehat{f}(\lambda)/\widehat{g}(\lambda) - 1| &= |\widehat{f}(\lambda) - \widehat{g(\lambda)}|/\widehat{g}(\lambda) \\ &= \frac{\left|\int_\Gamma (f(x) - g(x))e^{-(\lambda,x)}dx\right|}{\widehat{g}(\lambda)} \\ &\le \frac{\int_{|x|\le a}(f(x) + g(x))e^{-(\lambda,x)}dx}{\widehat{g}(\lambda)} + \varepsilon \frac{\int_{|x|>a} g(x)e^{-(\lambda,x)}dx}{\widehat{g}(\lambda)}. \end{aligned}$$

Since by the hypothesis $\widehat{g}(\lambda) \to \infty$ as $\lambda \to 0$, hence we obtain

$$\limsup_{\lambda\to 0,\ \lambda\in C} |\widehat{f}(\lambda)\widehat{g}(\lambda) - 1| \le \varepsilon,$$

which proves the theorem for $c > 0$ because ε is arbitrary. The proof for $c = 0$ is even more simple. □

We write $a(t) \asymp b(t)$ as $t \to \infty$ if

$$0 < \liminf_{t\to\infty} \left|\frac{a(t)}{b(t)}\right| \le \limsup_{t\to\infty} \left|\frac{a(t)}{b(t)}\right| < \infty.$$

The following theorem compares, roughly speaking, the asymptotic behaviour of a function and its Laplace transform.

THEOREM 1.5.6. *Let a function $f(x)$ be weakly oscillating in Γ at infinity, and* $\mathrm{ind}_- f > -n$. *Then for all $\lambda \in C$ and $e \in G$ as $t \to \infty$*

$$\widehat{f}(\lambda/t) \asymp t^n f(te) \asymp \int_{\Gamma(te)} f(x)\,dx,$$

where $\Gamma(x) = \{y: y \overset{\Gamma}{\leq} x\}$.

PROOF. Without loss of generality we assume that $|e| = 1$ (Theorem 1.4.5). Since $\mathrm{ind}_- f > -n$, by virtue of Corollary 1.4.4 there exist $\alpha, \beta > -n$, $b > 0$, $c > 0$, $c_0 > 0$ such that inequalities (1.4.25) and (1.4.26) hold for $t \geq b$. From these inequalities it follows that for $t \geq b$

$$\int_{A_t} \frac{f(tx)}{f(te)} e^{-(\lambda,x)} dx \leq \int_{A_t} e^{-(\lambda,x)} h(x)\,dx, \tag{1.5.33}$$

$$\int_{A_t} \frac{f(tx)}{f(te)} e^{-(\lambda,x)} dx \geq \int_{A_t} e^{-(\lambda,x)} g(x)\,dx, \tag{1.5.34}$$

where $A_t = \{x: x \in \Gamma,\ |x| \geq b/t\}$,

$$h(x) = \begin{cases} c|x|^\beta, & |x| \geq 1, \\ c|x|^\alpha, & b/t \leq |x| \leq 1, \end{cases} \qquad g(x) = \begin{cases} c_0|x|^\alpha, & |x| \geq 1, \\ c_0|x|^\beta, & b/t \leq |x| \leq 1. \end{cases}$$

Since $\mathrm{ind}_- f > -n$, we see that $t^n f(te) \to \infty$ as $t \to \infty$. Therefore,

$$\begin{aligned} \frac{\widehat{f}(\lambda/t)}{t^n f(te)} &= \int_\Gamma \frac{f(tx)}{f(te)} e^{-(\lambda,x)} dx = \int_{A_t} \frac{f(tx)}{f(te)} e^{-(\lambda,x)} dx + \int_{|y| \leq b} \frac{f(y)}{t^n f(te)} e^{-(\lambda/t,y)} dy \\ &= \int_{A_t} \frac{f(tx)}{f(te)} e^{-(\lambda,x)} dx + o(1), \qquad t \to \infty. \end{aligned} \tag{1.5.35}$$

From (1.5.33), (1.5.34), and (1.5.35) it follows that $\widehat{f}(\lambda/t) \asymp t^n f(te)$ as $t \to \infty$. Similar reasoning yields

$$t^n f(te) \asymp \int_{\Gamma(te)} f(x)\,dx, \qquad t \to \infty.$$

□

Let us give one more Tauberian theorem for double sequences which will be used below in studies of some classes of random permutations. Other results in this field can be found in (Alpár, 1976; Alpár, 1984; Omey, 1989; Omey, Willekens, 1989).

THEOREM 1.5.7. *For all $u, v \in (0,1)$, let the function*

$$A(u,v) = \sum_{m,n \geq 0} m^{\alpha-1} a(m,n) u^m v^n$$

be finite, $\alpha > 1$, $a(m,n) \geq 0$, *and*

$$A(e^{-\lambda/t}, e^{-\mu/t})/r(t) \to \lambda^{-\alpha} \mu^{-\gamma} \Gamma(\alpha)\Gamma(\gamma) \tag{1.5.36}$$

as $t \to \infty$ for any $\lambda, \mu > 0$, where $\gamma > 0$, $r(t)$ is some positive function of variable t, $\Gamma(\cdot)$ is the Euler gamma function. If $a(m,n)$ is monotone in m and

$$a(m,n) - a(m,l) = o\left(\sum_{i=0}^{l} a(m,i)/n\right) \tag{1.5.37}$$

as $n \to \infty$, $m \asymp n$, $l \geq n$, $l - n = o(n)$, then

$$a(tx,ty) \sim r(t)t^{-1-\alpha} y^{\gamma-1}$$

as $t \to \infty$ for any $x, y > 0$; for non-integer u, v, we set $a(u,v) = a([u],[v])$.

PROOF. As follows from (1.5.36) and Theorem 1.3.3, for any $x, y > 0$ as $t \to \infty$

$$\sum_{i=0}^{[tx]} \sum_{j=0}^{[ty]} i^{\alpha-1} a(i,j) \sim r(t) x^{\alpha} y^{\gamma}/\alpha\gamma. \tag{1.5.38}$$

Therefore, for any $\varepsilon > 0$, $x > 0$, $y > 0$ as $t \to \infty$

$$\frac{1}{r(t)} \sum_{i=[tx]+1}^{[tx(1+\varepsilon)]} \sum_{j=0}^{[ty]} i^{\alpha-1} a(i,j) \to \frac{x^{\alpha} y^{\gamma}}{\alpha\gamma}((1+\varepsilon)^{\alpha} - 1). \tag{1.5.39}$$

For the sake of definiteness, we assume that $a(m,n)$ does not increase in m. The expression in the left-hand side of (1.5.39) does not exceed

$$\frac{[tx(1+\varepsilon)] - [tx]}{r(t)} (tx(1+\varepsilon))^{\alpha-1} \sum_{j=0}^{[ty]} a(m,j),$$

where $m = [tx]$. Therefore,

$$\liminf_{t\to\infty} \frac{[tx(1+\varepsilon)] - [tx]}{r(t)} (tx(1+\varepsilon))^{\alpha-1} \sum_{j=0}^{[ty]} a(m,j) \geq \frac{x^{\alpha} y^{\gamma}}{\alpha\gamma}((1+\varepsilon)^{\alpha} - 1). \tag{1.5.40}$$

Since, as $t \to \infty$,

$$([tx(1+\varepsilon)] - [tx])(tx(1+\varepsilon))^{\alpha-1} \sim tx\varepsilon(tx(1+\varepsilon))^{\alpha-1} \sim \varepsilon(tx)^{\alpha}(1+\varepsilon)^{\alpha-1},$$

from (1.5.40) we obtain

$$\liminf_{t\to\infty} \sum_{j=0}^{[ty]} a(m,j) t^{\alpha}/r(t) \geq \frac{y^{\gamma}}{\alpha\gamma} \frac{(1+\varepsilon)^{\alpha} - 1}{\varepsilon(1+\varepsilon)^{\alpha-1}}.$$

If ε in the right-hand side of the obtained inequality tends to zero, we find that

$$\liminf_{t\to\infty} \sum_{j=0}^{[ty]} a(m,j) t^{\alpha}/r(t) \geq y^{\gamma}/\gamma.$$

Summing over i from $[tx(1-\varepsilon)]$ to $[tx]$, we similarly find that

$$\limsup_{t\to\infty}\sum_{j=0}^{[ty]} a(m,j)t^{\alpha}/r(t) \le y^{\gamma}/\gamma.$$

From the last two inequalities it follows that

$$\sum_{j=0}^{[ty]} a(m,j)t^{\alpha}/r(t) \to y^{\gamma}/\gamma, \qquad t\to\infty, \tag{1.5.41}$$

which yields for any $\delta\in(0,1)$

$$\sum_{j=[ty(1-\delta)]}^{[ty]} a(m,j)t^{\alpha}/r(t) \to \frac{y^{\gamma}}{\gamma}(1-(1-\delta)^{\gamma}). \tag{1.5.42}$$

We take an arbitrary $\varepsilon>0$ and choose δ, t_0 in such a way that

$$|a(m,j)-a(m,n)| \le \varepsilon\sum_{k=0}^{n} a(m,k)/n$$

for $t\ge t_0$ and $|j-[ty]) \le \delta ty+1$ where $n=[ty]$. Then

$$a(m,n) \le a(m,j)+\varepsilon\sum_{k=0}^{n} a(m,k)/n$$

for all $j\colon [ty(1-\delta)] \le j \le [ty]$ and $t\ge t_0$; hence for $M=[ty]-[ty(1-\delta)]$ we obtain

$$a(m,n)M \le \sum_{j=[ty(1-\delta)}^{n} a(m,j)+\frac{\varepsilon M}{n}\sum_{j=0}^{n} a(m,j),$$

or

$$a(m,n) \le \frac{1}{M}\sum_{j=[ty(1-\delta)]}^{n} a(m,j)+\frac{\varepsilon}{n}\sum_{j=0}^{n} a(m,j).$$

With the use of (1.5.41), (1.5.42), hence we obtain

$$\limsup_{t\to\infty}\frac{a(m,n)t^{1+\alpha}}{r(t)} \le \frac{1}{y\delta}\frac{y^{\gamma}}{\gamma}(1-(1-\delta)^{\gamma})+\frac{\varepsilon}{y}\frac{y^{\gamma}}{\gamma}.$$

Since ε, δ are arbitrary, we find that

$$\limsup_{t\to\infty}\frac{a(m,n)t^{1+\alpha}}{r(t)} \le y^{\gamma-1}.$$

Using similar lower bounds, we see that there exists

$$\lim_{t\to\infty}\frac{a(m,n)t^{1+\alpha}}{r(t)} = y^{\gamma-1}.$$

The theorem is thus proved. □

THEOREM 1.5.8. *Let the function $A(u, v)$ be the same as in Theorem* 1.5.7*, let* (1.5.36) *hold, $a(m, n)$ be monotone in m, and let*

$$a(m, n) - a(m, l) = O\left(\sum_{i=0}^{l} a(m, i)/n\right)$$

as $n \to \infty$, $m \asymp n$, $l \geq n$, $l - n = o(n)$. Then

$$a(m, n) = O(r(n)/n^{1+\alpha})$$

as $n \to \infty$ and $m \asymp n$.

The proof of this theorem repeats the above reasoning word for word.

While studying branching processes, we will also use the following Tauberian theorem.

THEOREM 1.5.9. *Let a function $r(t)$ be regularly varying at infinity with index $\gamma > -n$, a function $f(x)$ be measurable and non-negative in Γ and r-slowly varying at infinity in G (see Definition* 1.4.5*), and let $\widehat{f}(\lambda) < \infty$ for all $\lambda \in C$. If*

$$\widehat{f}(\lambda/t)/t^n r(t) \to \psi(\lambda) < \infty$$

as $t \to \infty$ for some solid cone $C_0 \prec C$ and all $\lambda \in C_0$, then

$$f(tx)/r(t) \to \varphi(x) < \infty$$

as $t \to \infty$ for all $x \in G$. Furthermore, there exists a measure Φ on Γ such that φ is its density in G and $\widetilde{\Phi}(\lambda) = \psi(\lambda)$, $\forall \lambda \in C$.

The proof of this theorem repeats the proof of the third assertion of Theorem 1.3.4 word for word.

1.6. One-dimensional Tauberian theorems

Let functions $f(t)$ and $g(t)$ be defined and positive for $t \geq a \geq 0$. We write

$$f(t) \overset{w}{\sim} g(t) \tag{1.6.1}$$

as $t \to \infty$ if for any ε there exists $\delta_0 \in (0, 1)$ such that for any $\delta \in (0, \delta_0)$ there is $t_0 > 0$ such that for $t \geq t_0$ the inequalities

$$(1 - \varepsilon)g(t(1 + \delta)) \leq f(t) \leq (1 + \varepsilon)g(t(1 - \delta))$$

hold. In the case where (1.6.1) holds, we say that the functions f and g are *weakly equivalent* at infinity. It is clear that if $g(t)$ is weakly oscillating at infinity (see Definition 1.4.4), then (1.6.1) implies the ordinary equivalence of $f(t)$ and $g(t)$ at infinity:

$$f(t) \sim g(t), \qquad t \to \infty,$$

that is, $f(t)/g(t) \to 1$ as $t \to \infty$.

First we prove the following Tauberian theorem.

THEOREM 1.6.1. *For $t > 0$, let functions $a_i(t) > 0$, $b_i(t) > 0$, $i = 1, 2$, be given, let $a_1(t)$, $b_1(t)$ do not increase, and $a_2(t)$, $b_2(t)$ be weakly oscillating at infinity. Let*

$$a(t) = a_1(t)a_2(t), \qquad b(t) = b_1(t)b_2(t),$$
$$A(t) = \int_0^t a(u)\,du \quad B(t) = \int_0^t b(u)\,du, \quad t > 0.$$

If $\widehat{a}(t) \sim \widehat{b}(t)$ as $t \to 0_+$, and for all $\lambda \in (0, 1)$

$$\limsup_{t\to\infty} B(\lambda t)/B(t) < 1, \tag{1.6.2}$$

then

$$a(t) \overset{w}{\sim} b(t), \qquad t \to \infty.$$

This Tauberian theorem, as well as two below, can be referred to as comparison theorems, because they compare the asymptotic behaviour at infinity of two functions and their Laplace transforms. Tauberian theorems of such type were also proved in (Stadtmüller, Trautner, 1979; Stadtmüller, Trautner, 1981; Stadtmüller, 1983; Omey, 1985b; Mikhalin, 1985). The proof of Theorem 1.6.1 is based on the following two lemmas.

LEMMA 1.6.1. *Let a function $f(t)$ defined for $t \geq 0$ be weakly oscillating at infinity and locally integrable on* $[0, \infty)$. *Then the function*

$$F(t) = \int_0^t f(u)\,du$$

is weakly oscillating at infinity as well.

LEMMA 1.6.2. *Let*

$$f(t) = \int_0^t g(u)\,dh(u),$$

where the function $g(t) > 0$ does not increase, and let the function $h(t) > 0$ do not decrease and be weakly oscillating at infinity. Then $f(t)$ is weakly oscillating at infinity.

PROOF OF LEMMA 1.6.1. For $t \geq \Delta > 0$ the inequalities

$$0 \leq \frac{\int_0^{t+\Delta} f(u)\,du - \int_0^t f(u)\,du}{\int_0^t f(u)\,du} \leq \frac{\int_t^{t+\Delta} f(u)\,du}{\int_{t/2}^t f(u)\,du}$$
$$\leq \frac{\Delta}{t/2} \frac{\sup_{u\in[t,2t]} f(u)/f(t)}{\inf_{u\in[t/2,t]} f(u)/f(t)} \to 0$$

are true as $t \to \infty$, and $\Delta = o(t)$ by virtue of Theorem 1.4.5. The lemma is proved. □

PROOF OF LEMMA 1.6.2. Let $\Delta > 0$. Then the inequalities

$$0 \leq \frac{f(t+\Delta) - f(t)}{f(t)} = \frac{\int_t^{t+\Delta} g(u)\,dh(u)}{\int_0^t g(u)\,dh(u)}$$
$$\leq \frac{g(t)\int_t^{t+\Delta} dh(u)}{g(t)\int_0^t dh(u)} = \frac{h(t+\Delta) - h(t)}{h(t) - h(0)} = o(1)$$

are true as $t \to \infty$, and $\Delta/t \to 0$. The lemma is proved. □

PROOF OF THEOREM 1.6.1. By virtue of Lemmas 1.6.1 and 1.6.2, the function $B(t)$ is weakly oscillating at infinity. Therefore, by virtue of Theorem 1.5.4, as $t \to \infty$

$$A(t) \sim B(t). \tag{1.6.3}$$

We fix an arbitrary $\lambda \in (0, 1)$. By virtue of (1.6.2), there are $t_0, c_0 > 0$ such that for $t \geq t_0$

$$\begin{aligned} \left| \frac{A(t) - A(\lambda t)}{B(t) - B(\lambda t)} - 1 \right| &\leq c_0 \left| \frac{A(t) - B(t) + B(\lambda t) - A(\lambda t)}{B(t)} \right| \\ &\leq c_0 \left| \frac{A(t)}{B(t)} - 1 \right| + c_0 \left| \frac{A(t\lambda)}{B(t\lambda)} - 1 \right| \frac{B(t\lambda)}{B(t)} = o(1) \end{aligned}$$

as $t \to \infty$ by virtue of (1.6.3). Therefore, for any $\lambda \in (0, 1)$ as $t \to \infty$

$$\int_{\lambda t}^{t} a(u)\, du \sim \int_{\lambda t}^{t} b(u)\, du. \tag{1.6.4}$$

We take an arbitrary $\varepsilon > 0$. Since the functions $a_2(t)$ and $b_2(t)$ are weakly oscillating at infinity, there are $t_1 > 0$ and $\delta_1 \in (0, 1)$ such that for all $t \geq t_1$, $\delta \in (0, \delta_1)$, and $u \in [ct, t]$ with $c = 1 - \delta$ the inequalities

$$\frac{a_2(u)}{a_2(t)} \geq \frac{1}{1 + \varepsilon_1}, \qquad \frac{b_2(u)}{b_2(ct)} \leq 1 + \varepsilon_1 \tag{1.6.5}$$

are true, where $\varepsilon_1 = (1 + \varepsilon)^{1/3} - 1$. We take an arbitrary $\delta \in (0, \delta_1)$. By virtue of (1.6.4), for it there exists $t_2 > t_1$ such that for $t \geq t_2$

$$\int_{ct}^{t} a(u)\, du \leq (1 + \varepsilon_1) \int_{ct}^{t} b(u)\, du.$$

Due to monotonicity of $a_1(t)$ and $b_1(t)$, hence it follows that for $t \geq t_2$

$$a_1(t) \int_{ct}^{t} a_2(u)\, du \leq (1 + \varepsilon_1) b_1(ct) \int_{ct}^{t} b_2(u)\, du.$$

From (1.6.5) and the last inequality we find that for $t \geq t_2$

$$\frac{a_1(t) a_2(t)}{1 + \varepsilon_1} \leq (1 + \varepsilon_1)^2 b_1(ct) b_2(ct),$$

hence

$$a(t) \leq (1 + \varepsilon) b((1 - \delta)t) \tag{1.6.6}$$

for $t \geq t_2$. Since $A(t) \sim B(t)$ as $t \to \infty$, by the same token for an arbitrary $\varepsilon_2 > 0$ there exists $\delta_2 \in (0, 1 - (1 + \delta_1)^{-1})$ such that for any $\delta_3 \in (0, \delta_2)$ there exists $t_3 > t_2$ such that for any $\tau \geq t_3$

$$b(\tau) \leq (1 + \varepsilon_2) a((1 - \delta_3)\tau). \tag{1.6.7}$$

In (1.6.7), let

$$\varepsilon_2 = (1-\varepsilon)^{-1}, \quad \delta_3 = 1-(1+\delta)^{-1}, \quad \tau = \frac{t}{1-\delta_3} = t(1+\delta).$$

Then (1.6.7) takes the form

$$b(t(1+\delta)) \le (1-\varepsilon)^{-1} a(t),$$

or, what is the same,

$$(1-\varepsilon) b(t(1+\delta)) \le a(t) \tag{1.6.8}$$

for $t \ge t_3(1-\delta_3) = t_3/(1+\delta)$. From (1.6.6) and (1.6.7) it follows that for $t \ge \max(t_2, t_3/(1+\delta))$ the inequalities

$$(1-\varepsilon) b(t(1+\delta)) \le a(t) \le (1+\varepsilon) b((1-\delta)t)$$

are true. The theorem is proved. □

As a corollary to Theorem 1.6.1, we derive the following Tauberian theorem.

THEOREM 1.6.2. *Let functions $f(t)$ and $g(t)$ be positive, not increasing,*

$$\limsup_{t\to\infty} g(t)/g(2t) < \infty, \tag{1.6.9}$$

and let there exist M such that for any fixed $n \ge M$

$$\frac{d^n}{d\lambda^n} \widehat{f}(\lambda) = (1+o(1)) \frac{d^n}{d\lambda^n} \widehat{g}(\lambda), \qquad \lambda \downarrow 0. \tag{1.6.10}$$

Then

$$f(t) \overset{w}{\sim} g(t), \qquad t \to \infty.$$

REMARK 1.6.1. In view of monotonicity of $q(t)$, from (1.6.9) it follows that for all $\lambda > 0$

$$0 < \liminf_{t\to\infty} \frac{q(\lambda t)}{q(t)} \le \limsup_{t\to\infty} \frac{q(\lambda t)}{q(t)} < \infty,$$

that is, the function $q(t)$ is *RO*-varying at infinity (see the beginning of the proof of Theorem 1.4.6). The monotone functions which possess this property are known as *dominatedly varying* at infinity (Seneta, 1976, A.3).

PROOF. By virtue of (1.6.9) and Theorem A.2 in (Seneta, 1976), there exists $n \ge M$ such that

$$\liminf_{t\to\infty} \frac{t^{n+1} g(t)}{\int_0^t u^n g(u)\, du} > 0. \tag{1.6.11}$$

For this n, we set

$$a(t) = t^n f(t), \quad b(t) = t^n g(t), \quad B(t) = \int_0^t b(u)\, du.$$

For any fixed $c \in (0, 1)$,

$$\frac{B(t) - B(ct)}{B(t)} = \frac{\int_{ct}^{t} u^n g(u)\, du}{\int_0^t u^n g(u)\, du} \geq \frac{g(t) \int_{ct}^{t} u^n du}{\int_0^t u^n g(u)\, du}.$$

From (1.6.11) and the last inequalities we obtain

$$\liminf_{t\to\infty} \frac{B(t) - B(ct)}{B(t)} > 0,$$

which yields (1.6.2). Therefore, by (1.6.10), all hypotheses of Theorem 1.6.1 are satisfied. By virtue of this theorem,

$$t^n f(t) \overset{w}{\sim} t^n g(t), \qquad t \to \infty,$$

hence we obtain

$$f(t) \overset{w}{\sim} g(t), \qquad t \to \infty.$$

The theorem is proved. □

THEOREM 1.6.3. *Let a function $g(t)$ do not increase, let inequality* (1.6.9) *hold, let a function $f(t)$ be differentiable for sufficiently large t, and*

$$f'(t) = O(g(t)/t), \qquad t \to \infty. \tag{1.6.12}$$

If there exists M such that for any fixed $n \geq M$

$$\frac{d^n}{d\lambda^n} \widehat{f}(\lambda) = o\left(\left|\frac{d^n}{d\lambda^n} \widehat{g}(\lambda)\right|\right), \qquad \lambda \downarrow 0, \tag{1.6.13}$$

then as $t \to \infty$

$$f(t) = o(g(t)).$$

PROOF. Let $n > -\operatorname{ind}_- g - 1$ (see Remark 1.4.4). By virtue of Theorem 1.5.6,

$$\widehat{T}_n g(1/t) \asymp t^{n+1} g(t), \qquad t \to \infty,$$

where $T_n g(t) = t^n g(t)$. Hence, by (1.6.13),

$$\widehat{T}_n f(1/t) = o(t^{n+1} g(t)), \qquad t \to \infty,$$

which is equivalent to the relation

$$\int_0^\infty e^{-x} x^n f(tx)\, dx = o(g(t)), \qquad t \to \infty.$$

Since $t^{n+1} g(t) \to \infty$ as $t \to \infty$), for any $b > 0$ we see that

$$I_1 = \int_{b/t}^\infty e^{-x} x^n \frac{f(tx)}{g(t)} dx = o(1), \qquad t \to \infty.$$

By the theorem of mean value, according to (1.6.12) $b > 0$ can be chosen so that for $t \geq b$ the inequalities

$$|f(tx) - f(t)| \leq c_0(x-1)g(t), \qquad x \geq 1,$$
$$|f(tx) - f(t)| \leq c_0(1-x)g(tx)/x, \qquad b/t \leq x \leq 1,$$

hold, where c_0 is a constant. By these inequalities,

$$|I_2| \leq c_0 \left(\int_1^\infty e^{-x} x^n (x-1)\, dx + \int_{b/t}^1 e^{-x} x^{n-1} \frac{g(tx)}{g(t)} dx \right),$$

where

$$I_2 = \int_{b/t}^\infty e^{-x} x^n \frac{f(tx) - f(t)}{g(t)} dx.$$

Hence it follows that $I_2 = O(1)$ as $t \to \infty$. Therefore,

$$\int_{b/t}^\infty e^{-x} x^n \frac{f(t)}{g(t)} dx = I_1 - I_2 = O(1), \qquad t \to \infty,$$

hence we obtain $f(t) = O(g(t))$ as $t \to \infty$. Since $n + \operatorname{ind}_- g > 0$, $b > 0$ can be chosen so that for $t \geq b$

$$f(t) \leq cg(t) \tag{1.6.14}$$

and

$$\frac{g(tx)}{g(t)} \leq c_1 x^\beta, \qquad b/t \leq x \leq 1, \tag{1.6.15}$$

where c, c_1 are some positive constants and $\beta + n > 0$ (see Corollary 1.4.4 and Remark 1.4.4). We consider the family of functions $f_t(x) = x^n f(tx)/g(t)$, $x \geq b/t$, $f_t(x) = 0$, $x < b/t$, for $t \geq b$. We check whether the hypotheses of Theorem 1.5.2 are satisfied or not.

1. As $t \to \infty$, $x_t \to x > 0$,

$$\begin{aligned} f_t(x_t) - f_t(x) &= x_t^n f(tx_t)/g(t) - x^n f(tx)/g(t) \\ &= \frac{f(tx_t)}{g(t)}(x_t^n - x^n) + \frac{x^n}{g(t)}(f(tx_t) - f(tx)) \\ &= \frac{O(g(tx_t))}{g(t)} o(1) + \frac{O(1)}{g(t)}(tx_t - tx) O(g(t)/t) = o(1) \end{aligned}$$

by (1.6.9), (1.6.12), and (1.6.14). The family of functions $\{f_t(x), t \geq b\}$ is hence asymptotically continuous as $t \to \infty$.

2. By (1.6.14) and (1.6.15), for $t \geq b$

$$f_t(x) \leq cx^n g(tx)/g(t) \leq \varphi(x),$$

where

$$\varphi(x) = \begin{cases} c_1 c x^n, & x \geq 1, \\ c_1 c x^{n+\beta}, & b/t \leq x \leq 1. \end{cases}$$

Since $n + \beta > 0$, $\widehat{\varphi}(\lambda) < \infty$ for any $\lambda > 0$.

3. Since $t^{n+1} g(t) \to \infty$ as $t \to \infty$, by (1.6.13)

$$\begin{aligned} \widehat{f}_t(\lambda) &= \frac{\int_b^\infty x^n f(tx) e^{-\lambda x}\, dx}{g(t)} = \frac{\int_{b/t}^\infty f(u) u^n e^{-u\lambda/t}\, du}{t^{n+1} g(t)} \\ &= \frac{\widehat{T}_n f(\lambda/t) + O(1)}{t^{n+1} g(t)} = o(1) + o(1) \frac{\widehat{T}_n g(\lambda/t)}{t^{n+1} g(t)} = o(1) \end{aligned}$$

as $t \to \infty$ for any fixed $\lambda > 0$. The hypotheses of Theorem 1.5.2 are thus satisfied, and the proof is complete. □

REMARK 1.6.2. As follows from the proof, in order for Theorem 1.6.3 be true it is sufficient that (1.6.12) is true and (1.6.13) is true for some $n > -\operatorname{ind}_- g$.

The Tauberian theorem below extends the well-known Tauberian theorem for power series (Feller, 1966, Section XIII.5, Theorem 5). A Tauberian theorem for asymptotic expansions of generating functions is obtained in (Vatutin, 1977c).

THEOREM 1.6.4. *Let a sequence $q_n \geq 0$ do not increase and*

$$g(s) = \sum_{k=0}^{\infty} q_k s^k.$$

If

$$\frac{d^m}{ds^m} g(s) \sim (1-s)^{-\alpha} L\left(\frac{1}{1-s}\right), \qquad s \uparrow 1, \tag{1.6.16}$$

for some $m \in \mathbf{N}$, $\alpha > 0$, and a slowly varying at infinity function $L(t)$, then, as $n \to \infty$,

$$q_n \sim n^{\alpha - m - 1} L(n) / \Gamma(\alpha). \tag{1.6.17}$$

PROOF. From (1.6.16) and Theorem 5 in (Feller, 1966, Section XIII.5) it follows that, as $n \to \infty$,

$$\sum_{k=0}^{n} k^{]m[} q_k \sim n^\alpha L(n) / \Gamma(\alpha + 1), \tag{1.6.18}$$

where $k^{]m[} = k(k-1) \cdots (k-m+1)$. Hence it follows that for any fixed $\lambda \in (0,1)$ as $n \to \infty$

$$\sum_{k=[\lambda n]+1}^{n} k^{]m[} q_k \sim (1 - \lambda^\alpha) n^\alpha L(n) / \Gamma(\alpha + 1). \tag{1.6.19}$$

From (1.6.19) and the inequality

$$[\lambda n]^{]m[} q_n (n - [\lambda n]) \le \sum_{k=[\lambda n]+1}^{n} k^{]m[} q_k$$

we obtain

$$\limsup_{n\to\infty} \frac{n^{m+1} q_n}{n^\alpha L(n)} \le \frac{1-\lambda^\alpha}{\Gamma(\alpha+1)} \limsup_{n\to\infty} \frac{n^{m+1}}{[\lambda n]^{]m[}(n-[\lambda n])} = \frac{1-\lambda^\alpha}{\Gamma(\alpha+1)\lambda^m \delta},$$

where $\delta = 1 - \lambda$. We observe that

$$\lim_{\delta\downarrow 0} \frac{1-\lambda^\alpha}{\lambda^m \delta} = \lim_{\delta\downarrow 0} \frac{1-(1-\delta)^\alpha}{\delta} = \alpha.$$

Hence

$$\limsup_{n\to\infty} \frac{n^{m+1} q_n}{n^\alpha L(n)} \le \frac{\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha)}. \tag{1.6.20}$$

From (1.6.18) it follows that for fixed $\lambda > 1$, as $n \to \infty$,

$$\sum_{k=n+1}^{[\lambda n]} k^{]m[} q_k \sim (\lambda^\alpha - 1) n^\alpha L(n) / \Gamma(\alpha+1).$$

The inequality

$$\sum_{k=n+1}^{[\lambda n]} k^{]m[} q_k \le q_n [\lambda n]^{]m[} ([\lambda n] - n)$$

is true. Therefore,

$$\liminf_{n\to\infty} \frac{q_n n^{m+1}}{n^\alpha L(n)} \ge \frac{\lambda^\alpha - 1}{\Gamma(\alpha+1)} \liminf_{n\to\infty} \frac{n^{m+1}}{[\lambda n]^{]m[}([\lambda n]-n)} = \frac{\lambda^\alpha - 1}{\Gamma(\alpha+1)\lambda^m \delta},$$

where $\delta = \lambda - 1$. Upon passing in the last inequality to the limit as $\delta \downarrow 0$, we obtain

$$\liminf_{n\to\infty} \frac{q_n n^{m+1}}{n^\alpha L(n)} \ge \frac{\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha)}. \tag{1.6.21}$$

From (1.6.20) and (1.6.21) it follows that there exists

$$\lim_{n\to\infty} \frac{q_n n^{m+1}}{n^\alpha L(n)} = \frac{1}{\Gamma(\alpha)},$$

which is equivalent to (1.6.17). The theorem is thus proved. □

THEOREM 1.6.5. *Let functions $a(t)$ and $b(t)$ be measurable and non-negative for $t \ge 0$; for $\lambda > 0$ let them possess the Laplace transforms $\widehat{a}(\lambda)$, $\widehat{b}(\lambda)$; let there be $s \ge 0$ such that the function $b(t)$ does not increase for $t \ge s$ and dominatedly varies at infinity:*

$$\limsup_{t\to\infty} \frac{b(t)}{b(2t)} < \infty, \tag{1.6.22}$$

for $y \ge x$, $y = x + o(x)$ let

$$\limsup_{x\to\infty} \frac{a(y) - a(x)}{b(x)} \le 0. \tag{1.6.23}$$

If there exists $M \in \mathbf{N}$ such that for any $n \ge M$ as $\lambda \downarrow 0$

$$\frac{d^n}{d\lambda^n}\widehat{a}(\lambda) = (1 + o(1))\frac{d^n}{d\lambda^n}\widehat{b}(\lambda), \tag{1.6.24}$$

then

$$a(t) \overset{w}{\sim} b(t), \qquad t \to \infty. \tag{1.6.25}$$

PROOF. Since the function $b(t)$ is dominatedly varying at infinity ((1.6.22) holds), by virtue of Theorem A.2 in (Seneta, 1976) there exists $M_1 \ge M$ such that for any fixed $n \ge M_1$

$$\limsup_{t\to\infty} \frac{\int_0^t u^n b(u)\,du}{t^{n+1}b(t)} < \infty. \tag{1.6.26}$$

Hence it follows that for such n and any $\lambda \in (0, 1)$

$$\limsup_{t\to\infty} \frac{B_n(\lambda t)}{B_n(t)} < 1, \tag{1.6.27}$$

where

$$B_n(t) = \int_0^t u^n b(u)\,du.$$

It is easily seen, indeed, that (1.6.27) is equivalent to the relation

$$\limsup_{t\to\infty} \frac{\int_{\lambda t}^t u^n b(u)\,du}{\int_0^t u^n b(u)\,du} > 0, \tag{1.6.28}$$

and (1.6.28), in its turn, follows from (1.6.26) because $b(t)$ does not increase. Furthermore, by virtue of Lemma 1.6.2, the function $B_n(t)$ is weakly oscillating at infinity (see Definition 1.4.4). Therefore, from Theorem 1.5.4 and relation (1.6.24) it follows that for any fixed $n \ge M$

$$A_n(t) = (1 + o(1))B_n(t), \qquad t \to \infty, \tag{1.6.29}$$

where

$$A_n(t) = \int_0^t u^n a(u)\,du.$$

We fix an arbitrary $\lambda \in (0, 1)$. In view of (1.6.27), there exist $t_0, c_0 > 0$ such that for $t \ge t_0$

$$\begin{aligned}\left|\frac{A_n(t) - A_n(t\lambda)}{B_n(t) - B_n(t\lambda)} - 1\right| &\le c_0 \frac{|A_n(t) - B_n(t) + B_n(t\lambda) - A_n(t\lambda)|}{B_n(t)} \\ &\le c_0\left|\frac{A_n(t)}{B_n(t)} - 1\right| + c_0\left|\frac{A_n(t\lambda)}{B_n(t\lambda)} - 1\right| \frac{B_n(t\lambda)}{B_n(t)} = o(1)\end{aligned}$$

as $t \to \infty$ by virtue of (1.6.29). So, for any fixed $\lambda \in (0,1)$ as $t \to \infty$

$$\int_{t\lambda}^{t} u^n a(u)\,du = (1+o(1)) \int_{t\lambda}^{t} u^n b(u)\,du. \tag{1.6.30}$$

We fix arbitrary $\varepsilon, \delta \in (0,1)$. By virtue of (1.6.22), there exist $x_0 \geq 2s$ and $c > 0$ such that for $x \geq x_0$

$$b(x) \leq cb(2x). \tag{1.6.31}$$

We set

$$\varepsilon_1 = \min\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{1+c}\right). \tag{1.6.32}$$

From (1.6.23) it follows that there exist $\delta_1 \in (0,\delta)$ and $x_1 \geq x_0$ such that for any $x \geq x_1$ and $y \in [x, x(1+\delta_1)]$ the inequality

$$a(y) - a(x) \leq \varepsilon_1 b(x) \tag{1.6.33}$$

is true. From (1.6.30) it follows that as $x \to \infty$

$$\int_{x}^{x(1+\delta_1)} u^n a(u)\,du = (1+o(1)) \int_{x}^{x(1+\delta_1)} u^n b(u)\,du. \tag{1.6.34}$$

According to (1.6.34), there exists $x_2 \geq x_1$ such that for any $x \geq x_2$

$$\int_{x}^{x(1+\delta_1)} y^n a(y)\,dy \geq (1-\varepsilon_1) \int_{x}^{x(1+\delta_1)} y^n b(y)\,dy. \tag{1.6.35}$$

From (1.6.33) and (1.6.35) it follows that for $x \geq x_2$

$$\int_{x}^{x(1+\delta_1)} (a(x)+\varepsilon_1 b(x)) y^n\,dy \geq \int_{x}^{x(1+\delta_1)} y^n a(y)\,dy \geq \int_{x}^{x(1+\delta_1)} y^n b(y)\,dy,$$

so that

$$(a(x)+\varepsilon_1 b(x))x^{n+1}\frac{(1+\delta_1)^{n+1}-1}{n+1} \geq (1-\varepsilon_1)x^{n+1}\frac{(1+\delta_1)^{n+1}-1}{n+1} b(x(1+\delta_1)),$$

hence we obtain

$$a(x)+\varepsilon_1 b(x) \geq b(x(1+\delta_1))(1-\varepsilon_1).$$

By virtue of (1.6.31) and monotonicity of $b(x)$, for $x \geq x_2$ the last inequality yields

$$a(x) + c\varepsilon_1 b(x(1+\delta_1)) \geq a(x) + \varepsilon_1 b(x) \geq (1-\varepsilon_1) b(x(1+\delta_1)),$$

hence it follows that

$$a(x) \geq (1-\varepsilon_1) b(x(1+\delta_1)) - c\varepsilon_1 b(x(1+\delta_1)),$$

then

$$a(x) \geq (1-(c+1)\varepsilon_1)b(x(1+\delta_1)) \qquad \forall x \geq x_2. \tag{1.6.36}$$

By virtue of (1.6.23), there exist $\delta_2 \in (0, \delta)$, $\delta_2 \leq 1/2$, and $x_3 \geq x_2$ such that for any $y \geq x_3$ and $x \in [y, y/(1-\delta_2)]$

$$a(x) - a(y) \leq \varepsilon_1 b(y),$$

which yields

$$a(y) \geq a(x) - \varepsilon_1 b(y) \geq a(x) - \varepsilon_1 b(x(1-\delta_2)). \tag{1.6.37}$$

In view of (1.6.30), there exists $x_4 \geq x_3$ such that for $x \geq x_4$

$$\int_{x(1-\delta_2)}^{x} y^n a(y)\,dy \leq (1+\varepsilon_1)\int_{x(1-\delta_2)}^{x} y^n b(y)\,dy. \tag{1.6.38}$$

From (1.6.37) and (1.6.38) we find that for $x \geq x_4$

$$\begin{aligned}\int_{x(1-\delta_2)}^{x} y^n (a(x) - \varepsilon_1 b((1-\delta_2)x))\,dy &\leq \int_{x(1-\delta_2)}^{x} y^n a(y)\,dy \\ &\leq (1+\varepsilon_1)\int_{x(1-\delta_2)}^{x} y^n b(y)\,dy,\end{aligned}$$

hence, due to monotonicity of $b(x)$, we obtain

$$a(x) - \varepsilon_1 b((1-\delta_2)x) \leq (1+\varepsilon_1)b((1-\delta_2)x),$$

which yields

$$a(x) \leq (1+2\varepsilon_1)b(x(1-\delta_2)) \tag{1.6.39}$$

for $x \geq x_4$. From relations (1.6.32), (1.6.36), and (1.6.39) it follows that for $x \geq x_4$

$$(1-\varepsilon)b(x(1+\delta_1)) \leq a(x) \leq (1+\varepsilon)b(x(1-\delta_2)). \tag{1.6.40}$$

We set $x_5 = \max(x_4, s/(1-\delta))$. Because of monotonicity of $b(x)$, (1.6.40) implies that for $x \geq x_5$

$$(1-\varepsilon)b(x(1+\delta)) \leq a(x) \leq (1+\varepsilon)b(x(1-\delta)).$$

The theorem is thus proved. □

Theorems 1.6.1, 1.6.2, 1.6.3, 1.6.5 are commonly referred to as comparison theorems (Vladimirov *et al.*, 1988) because they compare asymptotic behaviour of two functions. Tauberian theorems for dominatedly varying functions of ordinary type, that is, theorems where the asymptotic behaviour of a function is compared with that of its Laplace transform can be found in (Embrechts, 1978; de Haan, 1976; de Haan, Stadtmüller, 1985; Rogozin, 2002a; Rogozin, 2002b).

Let us prove one more Tauberian theorem for the Stieltjes transforms. Let a function $f(x)\colon \mathbf{R}_+ \to \mathbf{R}_+$, $R_+ = \{x\colon x \in \mathbf{R}^1,\ x > 0\}$, be weakly oscillating at infinity (see Definition 1.4.4). What this means in the case of one variable is that $f(y)/f(x) \to 1$ as $x \to \infty$, $y = x + o(x)$. By assertion 2 of Theorem 1.4.6, the function $f(x)$ for some $s > 0$ admits the representation

$$f(x) = \exp\left(\eta(x) + \int_s^x \frac{\varepsilon(t)}{t}dt\right), \qquad \forall x \geq s, \tag{1.6.41}$$

where the functions $\eta(x)$ and $\varepsilon(x)$ are measurable and bounded on $[s, \infty)$.

We fix some $l > 0$. The *Stieltjes transform* of the function $f(x)$ on $\mathbf{R}_+$ denoted $\check{f}(\lambda)$ is

$$\check{f}(\lambda) = \int_0^\infty \frac{f(x)\,dx}{(\lambda + x)^l}$$

(provided that the integral exists for $\lambda > 0$). Let us prove the following Tauberian theorem.

THEOREM 1.6.6. *Let a function $f(x)$ be weakly oscillating at infinity, and let $g(x) = r(x)h(x)$, $x \in \mathbf{R}_+$, where $r(x)$ is non-negative and monotone for $x \in R_+$ and the function $h(x)$ is weakly oscillating at infinity. We assume that some representation of $f(x)$ in the form* (1.6.41) *obeys the inequalities*

$$-1 < \inf_{x \geq s} \varepsilon(x) \leq \sup_{x \geq s} \varepsilon(x) < l - 1 \tag{1.6.42}$$

and, as $\lambda \to \infty$,

$$\check{g}(\lambda) = (1 + o(1))\check{f}(\lambda). \tag{1.6.43}$$

Then, as $x \to \infty$,

$$g(x) = (1 + o(1))f(x). \tag{1.6.44}$$

Theorem 1.6.6 generalises the corresponding assertions of (Keldysh, 1973; Matsaev, Palant, 1977); see also the book (Kostyuchenko, Sargsyan, 1979, Chapter X). The key difference of Theorem 1.6.6 from the preceding results consists of omitting the requirement that the functions $f(x)$ and $g(x)$ must be monotone. Theorem 1.6.6 is proved with the use of some assertions of Sections 1.4 and 1.5. As concerns other results in this field, we point out the book (Pilipović *et al.*, 1990) and papers (Belogrud, 1974; Sultanaev, 1974; Nikolić-Despotović, Pilipović, 1986; Selander, 1963; Stanković, 1985b).

In order to prove Theorem 1.6.6, we make use of two lemmas given below.

LEMMA 1.6.3. *Let a function $f(x)$ be weakly oscillating at infinity, and $g(x) = r(x)h(x)$, $x \in \mathbf{R}_+$, where $r(x)$ is non-negative and monotone for $x \in \mathbf{R}_+$, while the function $h(x)$ is weakly oscillating at infinity. We assume that some representation of $f(x)$ in the form* (1.6.41) *obeys the inequality*

$$\inf_{x \geq s} \varepsilon(x) > -1$$

and, as $\lambda \downarrow 0$,

$$\hat{g}(\lambda) = (1 + o(1))\hat{f}(\lambda).$$

Then, as $x \to \infty$,

$$g(x) = (1 + o(1))f(x).$$

Lemma 1.6.3 follows from Theorem 1.5.4.

We say that a function $a(x)$ defined in $\mathbf{R}_+$ is weakly oscillating at zero if the function $a(1/x)$ is weakly oscillating at infinity.

If a function $a(x)$ is weakly oscillating at zero, then from relation (1.6.41) it follows that the representation

$$a(x) = \exp\left(\omega(x) + \int_x^p \frac{\psi(u)}{u} du\right) \tag{1.6.45}$$

is true for all $x \in (0, p]$ with some $p > 0$, where the functions $\omega(x)$ and $\psi(x)$ are defined, bounded, and measurable for $x \in (0, p]$.

LEMMA 1.6.4. *Let* $a(x) = x^k a_1(x) > 0$, $b(x) = x^k b_1(x) > 0$, $x \in \mathbf{R}_+$, *where* $a_1(x)$ *and* $b_1(x)$ *are decreasing functions on* $\mathbf{R}_+$, $k > -1$, *let the function* $a(x)$ *be weakly oscillating at zero, and let some representation of* $a(x)$ *in the form* (1.6.45) *obey the inequality*

$$\sup_{0<u\le p} \psi(u) < 1.$$

If, as $y \to +\infty$,

$$\widehat{a}(y) = (1 + o(1))\widehat{b}(y), \tag{1.6.46}$$

then, as $x \downarrow 0$,

$$a(x) = (1 + o(1))b(x). \tag{1.6.47}$$

PROOF OF LEMMA 1.6.4. We assume the contrary, that is, for some sequence $\{x_m, m \in \mathbf{N}\}$, $x_m \downarrow 0$ as $m \to \infty$, let

$$\frac{b(x_m)}{a(x_m)} \to c \neq 1, \qquad c \in [0, \infty). \tag{1.6.48}$$

For $\lambda > 0$ we see that

$$\widehat{a}(\lambda/x_m) = \int_0^\infty e^{-\lambda y/x_m} a(y)\, dy = x_m a(x_m) \int_0^\infty \frac{a(x_m x)}{a(x_m)} e^{-\lambda x} dx. \tag{1.6.49}$$

We observe that the sequence

$$f_m(x) = \frac{a(x_m x)}{a(x_m)}, \qquad x \in \mathbf{R}_+,$$

is asymptotically continuous on $\mathbf{R}_+$. It is easy to see, indeed, that as $m \to \infty$, $y \to x > 0$

$$f_m(x) - f_m(y) = \frac{a(x_m x) - a(x_m y)}{a(x_m)} = \frac{a(x_m x)}{a(x_m)}\left(1 - \frac{a(x_m y)}{a(x_m x)}\right) = o(1),$$

because

$$1 - \frac{a(x_m y)}{a(x_m x)} \to 0, \quad m \to \infty, \quad y \to x > 0,$$

by the definition of a weakly oscillating function, and, as follows from representation (1.6.45),

$$\frac{a(x_m x)}{a(x_m)} = O(1)$$

as $m \to \infty$ provided $x > 0$ is fixed. Further, for $t \le p$, there exists a constant $c_1 < \infty$ such that by (1.6.45)

$$\begin{aligned}\frac{a(tx)}{a(t)} &= \exp\left(\omega(tx) - \omega(t) + \int_{tx}^{t} \frac{\psi(v)}{v} dv\right) \\ &\le \varphi(x) = \begin{cases} c_1 x^{\alpha}, & 1 \le x \le p/t, \\ c_1 x^{\beta}, & 0 < x \le 1, \end{cases}\end{aligned} \tag{1.6.50}$$

where

$$\alpha = \sup_{0<v\le p} (-\psi(v)), \qquad \beta = -\sup_{0<v\le p} \psi(v).$$

We set

$$g_m(x) = \begin{cases} f_m(x), & x \le p/x_m, \\ 0, & x > p/x_m. \end{cases}$$

Since the sequence $\{f_m(x),\ m \in \mathbf{N}\}$ is asymptotically continuous on $\mathbf{R}_+$, so is the sequence $\{g_m(x),\ m \in \mathbf{N}\}$. By virtue of Theorem 1.4.1, without loss of generality we can assume that for some continuous function $f(x)$ on $\mathbf{R}_+$

$$g_m(x) \to f(x), \qquad m \to \infty, \tag{1.6.51}$$

for any $x > 0$. From (1.6.50) and the definition of the functions $g_m(x)$ it follows that

$$g_m(x) \le \varphi(x) \tag{1.6.52}$$

for $m \in \mathbf{N}$ and $x > 0$. In addition,

$$\widehat{\varphi}(\lambda) < \infty \tag{1.6.53}$$

for all $\lambda > 0$ because by the hypothesis of the lemma

$$\beta = -\sup_{0<v\le p} \psi(v) > -1.$$

From (1.6.51), (1.6.52), and (1.6.53), with the use of the Lebesgue theorem we find that for all $\lambda > 0$

$$\widehat{g}_m(\lambda) \to \widehat{f}(\lambda) < \infty, \qquad m \to \infty. \tag{1.6.54}$$

Further, for $\lambda > 0$

$$\int_0^{\infty} \frac{a(x_m x)}{a(x_m)} e^{-\lambda x} dx = \widehat{g}_m(\lambda) + \frac{\int_p^{\infty} a(y) e^{-\lambda y/x_m} dy}{x_m a(x_m)}. \tag{1.6.55}$$

By (1.6.45), for $x \in (0, p]$ and some constant $c_2 > 0$

$$a(x) \geq c_2 x^{\gamma}, \tag{1.6.56}$$

where

$$\gamma = -\inf_{0<v\leq p} \psi(v).$$

From (1.6.56) it follows that

$$\begin{aligned}\frac{\int_p^\infty a(y)e^{-\lambda y/x_m}dy}{x_m a(x_m)} &\leq \frac{e^{-\lambda p/(2x_m)}\int_p^\infty a(y)e^{-\lambda y/(2x_m)}dy}{x_m a(x_m)} \\ &= O\left(\frac{e^{-\lambda p/(2x_m)}}{x_m a(x_m)}\right)o(1) = o(1)\end{aligned}$$

as $m \to \infty$. Taking (1.6.54) into account, from (1.6.55) we obtain

$$\int_0^\infty \frac{a(x_m x)}{a(x_m)} e^{-\lambda x} dx \to \widehat{f}(\lambda). \tag{1.6.57}$$

We observe that for $\lambda > 0$

$$\widehat{b}(\lambda/x_m) = x_m a(x_m) \int_0^\infty \frac{b(x_m x)}{a(x_m)} e^{-\lambda x} dx. \tag{1.6.58}$$

By virtue of (1.6.46), (1.6.49), (1.6.57), and (1.6.58), as $m \to \infty$ we obtain

$$\int_0^\infty \frac{b(x_m x)}{a(x_m)} e^{-\lambda x} dx \to \widehat{f}(\lambda) \qquad \forall \lambda > 0. \tag{1.6.59}$$

By virtue of the continuity theorem for Laplace transforms (Theorem 1.3.2),

$$\int_A \frac{b(x_m x)}{a(x_m)} dx \to \int_A f(x)dx \tag{1.6.60}$$

for any interval $A \subseteq \mathbf{R}_+$. We fix some $\delta \in (0, 1)$. Let us demonstrate that for some constant $c_3 < \infty$ and all $m \in \mathbf{N}$ the bound

$$\frac{b_1(x_m)}{a_1(x_m)} \leq c_3 \tag{1.6.61}$$

holds true. It is easy to see, indeed, that, by (1.6.60), there exists a constant $c_4 < \infty$ such that for all $m \in \mathbf{N}$

$$\int_{1-\delta}^1 \frac{b(x_m x)}{a(x_m)} dx = \int_{1-\delta}^1 \frac{b_1(x_m x)x^k}{a_1(x_m)} dx \leq c_4,$$

which implies that

$$\frac{b_1(x_m)}{a_1(x_m)} \int_{1-\delta}^1 x^k dx \leq c_4,$$

that is, (1.6.61) holds with

$$c_3 = c_4 \Big/ \int_{1-\delta}^{1} x^k \, dx \ .$$

We observe that

$$\begin{aligned}
\frac{b(x_m)}{a(x_m)} &= \frac{1}{\delta}\int_{1-\delta}^{1} \frac{b(x_m)}{a(x_m)} dx = \frac{1}{\delta}\left(\int_{1-\delta}^{1} \frac{b(x_m)-b(x_m x)}{a(x_m)} dx + \int_{1-\delta}^{1} \frac{b(x_m x)}{a(x_m)} dx\right) \\
&= \frac{1}{\delta}\left(\int_{1-\delta}^{1} \frac{b(x_m x)}{a(x_m)} dx + \int_{1-\delta}^{1} \frac{b_1(x_m) - b_1(x_m x) x^k}{a_1(x_m)} dx\right) \\
&\le \frac{1}{\delta}\left(\int_{1-\delta}^{1} \frac{b(x_m x)}{a(x_m)} dx + \frac{b_1(x_m)}{a_1(x_m)} \int_{1-\delta}^{1} (1-x^k)\, dx\right) \\
&= \frac{1}{\delta}\int_{1-\delta}^{1} \frac{b(x_m x)}{a(x_m)} dx + \frac{(k+1)\delta - (1-(1-\delta)^{k+1})}{(k+1)\delta} \frac{b_1(x_m)}{a_1(x_m)}.
\end{aligned} \tag{1.6.62}$$

From (1.6.60), (1.6.61), and (1.6.62) it follows that

$$\limsup_{m\to\infty} \frac{b(x_m)}{a(x_m)} \le \frac{1}{\delta}\int_{1-\delta}^{1} f(x)\, dx + c_3 \frac{(k+1)\delta - (1-(1-\delta)^{k+1})}{(k+1)\delta}.$$

If δ tends to zero in the right-hand side of this inequality, taking into account the continuity of $f(x)$ we obtain

$$\limsup_{m\to\infty} \frac{b(x_m)}{a(x_m)} \le f(1).$$

In the same way, with the use of integration from 1 to $1+\delta$, we arrive at

$$\liminf_{m\to\infty} \frac{b(x_m)}{a(x_m)} \ge f(1).$$

Thus, there exists

$$\lim_{m\to\infty} \frac{b(x_m)}{a(x_m)} = f(1).$$

Recalling (1.6.51) and the definition of the functions $g_m(x)$, we conclude that

$$f(1) = 1.$$

Thus,

$$\frac{b(x_m)}{a(x_m)} \to 1, \qquad m \to \infty.$$

The last relation contradicts (1.6.48), which proves the lemma. □

PROOF OF THEOREM 1.6.6. Let us prove that for some $p < 1/s$ and $\lambda \in (0, p]$ the transform $\widehat{f}(\lambda)$ admits the representation

$$\widehat{f}(\lambda) = \frac{1}{\lambda} \exp\left(\int_{1/p}^{1/\lambda} \frac{\varepsilon(t)}{t} dt + \xi(\lambda)\right), \tag{1.6.63}$$

where $\varepsilon(t)$ is the function from representation (1.6.41) of $f(x)$ and $\xi(\lambda)$ is bounded for $\lambda \in (0, 1/p]$. We supplement the definition of $\varepsilon(x)$ and $\eta(x)$ with identical zero for $x < s$. Since

$$\inf_{x \geq s} \varepsilon(x) > -1,$$

we see that $\widehat{f}(\lambda) \to \infty$ as $\lambda \downarrow 0$. Therefore, as $\lambda \downarrow 0$,

$$\begin{aligned}\widehat{f}(\lambda) &= \int_0^\infty e^{-\lambda x} \exp\left(\eta(x) + \int_0^x \frac{\varepsilon(t)}{t} dt\right) dx + O(1)\\ &= (1 + o(1)) \int_0^\infty e^{-\lambda x} \exp\left(\eta(x) + \int_0^x \frac{\varepsilon(t)}{t} dt\right) dx.\end{aligned}$$

So, it suffices to obtain a similar representation of

$$\int_0^\infty e^{-\lambda x} \exp\left(\eta(x) + \int_0^x \frac{\varepsilon(t)}{t} dt\right) dx.$$

We observe that

$$\begin{aligned}\int_0^\infty e^{-\lambda x} \exp\left(\eta(x) + \int_0^x \frac{\varepsilon(t)}{t} dt\right) dx &= \frac{1}{\lambda} \int_0^\infty e^{-u} \exp\left(\eta\left(\frac{u}{\lambda}\right) + \int_0^{u/\lambda} \frac{\varepsilon(t)}{t} dt\right) du\\ &= \frac{1}{\lambda} \exp\left(\int_0^{1/\lambda} \frac{\varepsilon(t)}{t} dt\right) I(\lambda),\end{aligned}$$

where

$$I(\lambda) = \int_0^\infty e^{-u} \exp\left(\eta\left(\frac{u}{\lambda}\right) + \int_{1/\lambda}^{u/\lambda} \frac{\varepsilon(t)}{t} dt\right) du.$$

In order to prove (1.6.63), it suffices to demonstrate that there exist constants c_1, c_2 such that

$$0 < c_1 \leq I(\lambda) \leq c_2 < \infty \tag{1.6.64}$$

for $\lambda \in (0, 1/s]$. We represent the integral $I(\lambda)$ as the sum of the integrals $I_1(\lambda)$ and $I_2(\lambda)$ over u from 0 to 1 and from 1 to ∞, respectively:

$$I(\lambda) = I_1(\lambda) + I_2(\lambda). \tag{1.6.65}$$

For $\lambda \in (0, 1/s]$ we obtain

$$I_1(\lambda) = \int_0^1 e^{-u} \exp\left(\eta\left(\frac{u}{\lambda}\right) - \int_{u/\lambda}^{1/\lambda} \frac{\varepsilon(t)}{t} dt\right) du \leq e^{c_3} \int_0^1 \exp(-\alpha(-\ln u))\, du,$$

where

$$c_3 = \sup_{v \geq 0} \eta(v), \qquad \alpha = \inf_{t \geq s} \varepsilon(t).$$

Therefore, for $\lambda \in (0, 1/s]$

$$I_1(\lambda) \le e^{c_3} \int_0^1 u^\alpha du = \frac{e^{c_3}}{\alpha + 1}, \tag{1.6.66}$$

because $\alpha > -1$ by inequalities (1.6.42). Furthermore,

$$\begin{aligned} I_2(\lambda) &= \int_1^\infty e^{-u} \exp\left(\eta\left(\frac{u}{\lambda}\right) + \int_{1/\lambda}^{u/\lambda} \frac{\varepsilon(t)}{t} dt\right) du \\ &\le e^{c_3} \int_1^\infty e^{-u} \exp(\beta \ln u) du = e^{c_3} \int_1^\infty e^{-u} u^\beta du < \infty, \end{aligned} \tag{1.6.67}$$

where

$$\beta = \sup_{t \ge s} \varepsilon(t).$$

In addition,

$$I_1(\lambda) \ge e^{c_4} \int_0^1 e^{-u} \exp(-\beta(-\ln u))\, du = e^{c_4} \int_0^1 e^{-u} u^\beta du > 0, \tag{1.6.68}$$

$$I_2(\lambda) \ge e^{c_4} \int_1^\infty e^{-u} \exp(\alpha \ln u)\, du = e^{c_4} \int_1^\infty e^{-u} u^\alpha du > 0, \tag{1.6.69}$$

where

$$c_4 = \inf_{v \ge 0} \eta(v).$$

Bound (1.6.64) follows from relations (1.6.65)–(1.6.69). Next, let us demonstrate that $\widehat{f}(\lambda)$ is weakly oscillating at zero. Let $\mu, \lambda \downarrow 0$, $\mu/\lambda \to 1$. Then

$$\begin{aligned} \widehat{f}(\lambda) &= \int_0^\infty e^{-\lambda x} f(x)\, dx = \frac{\mu}{\lambda} \int_0^\infty e^{-\mu u} f\left(\frac{\mu}{\lambda} u\right) du \\ &= (1 + o(1)) \int_0^\infty e^{-\mu u} f\left(\frac{\mu}{\lambda} u\right) du \\ &= (1 + o(1)) \left(\widehat{f}(\mu) + \int_0^\infty e^{-\mu u} f(u) \left(1 - \frac{f\left(\frac{\mu}{\lambda} u\right)}{f(u)}\right) du\right). \end{aligned}$$

So, it suffices to show that, as $\lambda, \mu \downarrow 0$, $\lambda/\mu \to 1$,

$$\int_0^\infty e^{-\mu u} f(u) \left(1 - \frac{f\left(\frac{\mu}{\lambda} u\right)}{f(u)}\right) du = o(\widehat{f}(\mu)) \tag{1.6.70}$$

First, we see that, as $\lambda, \mu \downarrow 0$, $\lambda/\mu \to 1$, for any fixed $t > 0$

$$\begin{aligned} \int_0^t e^{-\mu u} f(u) \left(1 - \frac{f\left(\frac{\mu}{\lambda} u\right)}{f(u)}\right) du &= \int_0^t e^{-\mu u} f(u) du - \int_0^t e^{-\mu u} f\left(\frac{\mu}{\lambda} u\right) du \\ &= O(1) = o(\widehat{f}(\mu)). \end{aligned} \tag{1.6.71}$$

We fix an arbitrary $\varepsilon > 0$, and for this ε choose $t > 0$ and $\delta > 0$ in such a way that the inequality

$$\left| 1 - \frac{f\left(\frac{\mu}{\lambda}u\right)}{f(u)} \right| \le \varepsilon \tag{1.6.72}$$

holds true for $u \ge t$ and $|\mu/\lambda - 1| \le \delta$. Then (1.6.71) and (1.6.72) yield

$$\limsup_{\mu\downarrow 0, \lambda/\mu\to 1} \frac{1}{\widehat{f}(\mu)} \left| \int_0^\infty e^{-\mu u} f(u) \left(1 - \frac{f\left(\frac{\mu}{\lambda}u\right)}{f(u)}\right) du = \right| \le \varepsilon.$$

Since ε is an arbitrary positive number, (1.6.70) now follows from the last inequality. As we know (Seneta, 1976, Theorem 2.5), a Stieltjes transform is a double Laplace transform, namely, for $\lambda > 0$

$$\breve{f}(\lambda) = \int_0^\infty a(x)e^{-x\lambda}dx, \qquad \breve{g}(\lambda) = \int_0^\infty b(x)e^{-x\lambda}dx, \tag{1.6.73}$$

where

$$a(x) = x^{l-1}\widehat{f}(x)/\Gamma(l), \qquad b(x) = x^{l-1}\widehat{g}(x)/\Gamma(l). \tag{1.6.74}$$

As we have seen, $\widehat{f}(x)$ weakly oscillates at zero, hence so does the function $a(x)$, and from (1.6.63) it now follows that for $x \in (0, 1/p]$

$$\begin{aligned} a(x) &= x^{l-2}\exp\left(\int_{1/p}^{1/x} \frac{\varepsilon(t)}{t}dt + \xi(x) - \ln\Gamma(l)\right) \\ &= x^{l-2}\exp\left(\int_x^p \frac{\varepsilon(1/u)}{u}du + \xi(x) - \ln\Gamma(l)\right) \\ &= \exp\left(\int_x^p \frac{\psi(u)}{u}du + \omega(x)\right), \end{aligned}$$

where

$$\psi(x) = 2 - l + \varepsilon(1/x), \qquad \omega(x) = \xi(x) - \ln\Gamma(l) + (2-l)\ln p.$$

By (1.6.42),

$$\sup_{0<x\le p} \psi(x) = 2 - l + \sup_{0<x\le p} \varepsilon(1/x) < 1.$$

Making use of Lemma 1.6.4, we see that, as $x \downarrow 0$,

$$a(x) = (1 + o(1))b(x),$$

which, with account for (1.6.74), implies that, as $x \downarrow 0$,

$$\widehat{f}(x) = (1 + o(1))\widehat{g}(x).$$

From the last relation and Lemma 1.6.3 it follows that, as $x \to \infty$,

$$f(x) = (1 + o(1))g(x),$$

which completes the proof of the theorem. □

1.7. Tauberian theorems of Drozhzhinov–Zavyalov type

In this section, using an n-faced cone as an example, we formulate a multidimensional Abelian theorem and three multidimensional Tauberian theorems proved by Yu. N. Drozhzhinov and B. I. Zavyalov. Such a contraction of the class of cones is due to two reasons. First, we believe that n-faced cones, particularly octants, will find widespread application in probability theory. Out of three probabilistic applications of multidimensional Tauberian theorems, two are obtained for n-faced cones (see Chapters 2 and 3). Only one of them concerns a wider class of cones (Chapter 4). Second, we do not want to introduce a new notation which would severely hamper the reading of this section. For an intrigued reader, we will speak about a wide class of cones, though, for which multidimensional Tauberian theorems of Drozhzhinov–Zavyalov type remain true. We also cite two one-dimensional Tauberian theorems of Drozhzhinov–Zavyalov type for asymptotic expansions of Laplace transforms and characteristic functions of random variables. We believe that these Tauberian theorems are of much interest to probabilists.

Let Γ be a closed *n-faced cone* in $\mathbf{R}^n$ with apex at zero, that is, let there exist a base $e_1, e_2, \dots, e_n \in \mathbf{R}^n$, $|e_i| = 1$, $i = 1, \dots, n$, such that

$$\Gamma = \{x : x \in \mathbf{R}^n,\ (x, e_k) \ge 0\ \forall k = 1, \dots, n\}.$$

We assume that $\{U_k,\ k \in I\}$ is a family of non-singular linear operators in $\mathbf{R}^n$ which leave invariant the cone Γ:

$$U_k\Gamma = \Gamma \quad \forall k \in I, \quad J_k = \det U_k.$$

We let $I \subset \mathbf{R}^1$ which has $+\infty$ as its limit point. The operator $V_k = (U_k^*)^{-1}$ leaves invariant the dual cone

$$\Gamma^* = \{y : y \in \mathbf{R}^n, (y, x) \ge 0\ \forall x \in \Gamma\}.$$

For an arbitrary operator U_k we obtain

$$\Lambda(k) = \sup_{|e|=1} |U_k e|, \qquad \lambda(k) = \inf_{|e|=1} |U_k e|. \tag{1.7.1}$$

We say that the family $\{U_k,\ k \in I\}$ is of

FIRST TYPE if $\Lambda(k) \to +\infty$ and $\lambda(k) \to +\infty$ as $k \to +\infty$;

SECOND TYPE if there exists $b > 0$ such that $\Lambda(k) \to +\infty$ as $k \to +\infty$, $\lambda(k) \ge b > 0$;

THIRD TYPE if $\Lambda(k) \to +\infty$ as $k \to +\infty$.

Let μ be a *tempered measure* on Γ, that is, for some $p \ge 0$

$$\int_\Gamma \frac{\mu(dx)}{1 + |x|^p} < \infty.$$

For $x \in \Gamma$, let $\mu(x) = \mu\{\Gamma \cap (x - \Gamma)\}$. As before, let $\widetilde{\mu}(y)$ denote the Laplace transform of the measure μ:

$$\widetilde{\mu}(y) = \int_\Gamma e^{-(x,y)} \mu(dx), \qquad y \in G = \operatorname{int} \Gamma.$$

DEFINITION 1.7.1. A measure μ on Γ is said to be an *admissible measure* of first (second, third) type for a cone Γ if it satisfies the following conditions:

(1) $\mu(x) > 0$ for $x \in G = \operatorname{int}\Gamma$;

(2) for any family U_k, $k \in I$, of linear operators of first (second, third) type which leave invariant the cone Γ, there exist $x^0 \in G$ and a subsequence $\{U_{k_m},\ k_m \to +\infty\}$ as $m \to +\infty$ such that

$$\frac{\mu(U_{k_m}x)}{\mu(U_{k_m}x^0)} \to g(x) > 0, \qquad x \in G,$$

as $m \to +\infty$ uniformly with respect to x in an arbitrary compact $Q \subset G$, where $g(x)$ is a continuous function for $x \in G$;

(3) there exists m_0 such that

$$\frac{\mu(U_{k_m}x)}{\mu(U_{k_m}x^0)} \le \psi(x), \qquad m > m_0,$$

and there exists q such that

$$\int \frac{\psi(x)}{1 + |x|^q} < \infty. \tag{1.7.2}$$

The following sufficient condition for admissibility of a measure μ is valid.

THEOREM 1.7.1. *Let $\mu(x)$ be positive, continuously differentiable in G, and obey the condition*

$$-1 < a \le (e_i, x)(e_i, \nabla\mu(x))/\mu(x) \le b, \quad x \in G, \quad i = 1, \dots, n,$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. Then the measure μ is an admissible measure of each type for Γ.

For $x \in \Gamma$, let $\Delta_\Gamma(x)$ stand for the distance between the vector x and the boundary of the cone Γ.

DEFINITION 1.7.2. Let $f(x)$ be a locally integrable function in Γ. We say that $f(x)$ *tends to* 1 *in the sense of the cone* Γ *as* $x \to \infty$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - 1| < \varepsilon \text{ for } \Delta_\Gamma(x) > \delta, \quad f(x) \to 1, \quad x \to \infty \text{ in } \Gamma.$$

The following Tauberian comparison theorem is true (see (Drozhzhinov, Zavyalov, 1984) or (Vladimirov *et al.*, 1988, Section II.6.2, Theorem 2)).

THEOREM 1.7.2. *Let μ and ν be non-negative tempered measures on* Γ.

(1) *If μ is an admissible measure of first type for the cone Γ and $\widetilde{\nu}(y)/\widetilde{\mu}(y) \to 1$ as $y \to 0$, $y \in C = \operatorname{int}\Gamma^*$, then*

$$\nu(x)/\mu(x) \to 1, \quad x \to \infty \textit{ in } \Gamma.$$

(2) *If* μ *is an admissible measure of second type for the cone* Γ *and for any* $b > 0$

$$\widetilde{\nu}(y)/\widetilde{\mu}(y) \to 1 \textit{ for } \Delta_C(y) \to 0, \quad y \in C, \quad |y| < b,$$

then for any $\delta > 0$

$$\nu(x)/\mu(x) \to 1 \textit{ for } |x| \to +\infty, \quad \Delta_\Gamma(x) > \delta.$$

(3) *If* μ *is an admissible measure of third type for the cone* Γ *and*

$$\widetilde{\nu}(y)/\widetilde{\mu}(y) \to 1 \textit{ for } \Delta_C(y) \to 0, \quad y \in C,$$

then

$$\nu(x)/\mu(x) \to 1 \textit{ for } |x| \to +\infty, \quad x \in G.$$

Let, as before, $\{U_k,\ k \in I\}$ be some family of non-singular linear operators which leave invariant the cone Γ, and let Λ_k, λ_k be defined by relation (1.7.1)

DEFINITION 1.7.3. A measure μ on Γ is said to be *completely admissible* for a family $\{U_k,\ k \in I\}$ if the following conditions are satisfied:

(1) $\mu(x) > 0$ for $x \in G = \operatorname{int} \Gamma$;

(2) there exists a vector $x^0 \in G$ such that

$$\frac{\mu(U_k x)}{\mu(U_k x^0)} \to g(x) > 0, \quad k \to +\infty, \quad k \in I,$$

uniformly with respect to x in any compact $Q \subset G$, where $g(x)$ is a continuous function in G;

(3) there exists k_0 such that

$$\frac{\mu(U_k x)}{\mu(U_k x^0)} \leq \psi(x), \quad k > k_0, \quad x \in G,$$

where $\psi(x)$ is a tempered function on Γ, that is, relation (1.7.2) holds.

In (Vladimirov *et al.*, 1988, Section II.6.1, Theorem 3), the following Tauberian comparison theorem is proved.

THEOREM 1.7.3. *Let* μ *and* ν *be non-negative tempered measures on* Γ*, and let the measure* μ *be completely admissible for a family* $\{U_k, k \in I\}$*. If the following conditions are satisfied:*

(1) *there exists an open set* $\Omega \subset C = \operatorname{int} \Gamma^*$ *such that*

$$\frac{\widetilde{\nu}(V_k y)}{\widetilde{\mu}(V_k y)} \to 1, \quad k \to \infty, \quad k \in I, \quad y \in \Omega, \quad V_k = (U_k^*)^{-1};$$

(2) *there exist numbers M, β, k_0, and a vector $e \in C$ such that*

$$\frac{\widetilde{\nu}(V_k \delta e)}{\widetilde{\mu}(V_k \delta e)} \le \frac{M}{\delta^\beta}, \quad 0 < \delta \le 1, \quad k > k_0, \quad k \in I;$$

then

$$\frac{\nu(U_k x)}{\mu(U_k x)} \to 1, \quad k \to \infty, \quad k \in I,$$

uniformly in $x \in K$ for any compact $K \subset G = \operatorname{int}\Gamma$.

DEFINITION 1.7.4. A measure μ on Γ is said to be *a-admissible* for a cone Γ, if for any family of non-singular linear operators $\{U_k, k \in I\}$ which leave invariant the cone Γ such that $\lambda_k \ge p\Lambda_k^a$ there exists a subsequence $\{U_{k_m}, m \to +\infty, k_m \in I\}$ for which μ is completely admissible.

Let $D_a(\Gamma)$ stand for the set of all a-admissible measures for a cone Γ. It is clear that $a \le 1$.

THEOREM 1.7.4. *Let a measure μ satisfy the hypotheses of Theorem* 1.7.1. *Then $\mu \in D_a(\Gamma)$ for any $a \le 1$.*

In (Drozhzhinov, Zavyalov, 1990), the following Tauberian comparison theorem is proved.

THEOREM 1.7.5. *Let μ and ν be non-negative tempered measures on Γ, and $\mu \in D_a(\Gamma)$. If for an arbitrary constant $c_1 > 0$*

$$\frac{\widetilde{\nu}(y)}{\widetilde{\mu}(y)} \to 1 \text{ for } \Delta_C(y) \to 0, \quad |y| \le c_1 \Delta_C^a(y), \quad y \in C,$$

then for an arbitrary constant $c_2 > 0$

$$\frac{\nu(x)}{\mu(x)} \to 1 \text{ for } |x| \to \infty, \quad \Delta_\Gamma(x) \ge c_2 |x|^a, \quad x \in G.$$

The appropriately altered Theorems 1.7.3 and 1.7.5 are true in the case where one of the measures is complex-valued (Drozhzhinov, Zavyalov, 1992, Theorems 6 and 7). In (Drozhzhinov, Zavyalov, 1990, Theorem 4), the following Abelian comparison theorem is given.

THEOREM 1.7.6. *Let μ and ν be non-negative tempered measures on Γ, and $\mu \in D_a(\Gamma)$. If for an arbitrary constant $c > 0$*

$$\frac{\nu(x)}{\mu(x)} \to 1 \text{ for } |x| \to \infty, \quad \Delta_\Gamma(x) \ge c |x|^a, \quad x \in G,$$

then for an arbitrary constant $c_1 > 0$

$$\frac{\widetilde{\nu}(y)}{\widetilde{\mu}(y)} \to 1 \text{ for } \Delta_C(y) \to 0, \quad |y| \le c_1 \Delta_C^a(y), \quad y \in C.$$

In (Drozhzhinov, Zavyalov, 1990, Theorem 3), an Abelian comparison theorem for tempered distributions was proved. The proof of that Abelian theorem came up against severe analytical difficulties. Additional constraints should be imposed on the functions under consideration naturally referred to as Abelian conditions. The corresponding quite complicated counterexample was considered there (Section 3).

In what follows, we consider a much more wide class of cones for which Tauberian theorems 1.7.2, 1.7.3, and 1.7.5 are true, as well as Abelian theorem 1.7.6.

DEFINITION 1.7.5. A convex cone Γ is said to be *homogeneous* if for any vectors a and b in $G = \operatorname{int}\Gamma$ there exists a non-singular linear operator U which leaves invariant the cone Γ such that $Ua = b$.

Let Γ be a closed convex acute solid cone in $\mathbf{R}^n$ with apex at zero (see the beginning of Section 1.1). As before, let Γ^* denote the cone dual to the cone Γ:

$$\Gamma^* = \{y: y \in \mathbf{R}^n, (y, x) \geq 0 \ \forall x \in \Gamma\},$$

$C = \operatorname{int}\Gamma^*$. Let $S'(\Gamma)$ stand for the space of tempered distributions with supports in Γ; it is dual to the space $S(\Gamma)$ of infinitely differentiable functions $\varphi(x)$ such that

$$p_m(\varphi) = \max_{|k|\leq m} \sup_{x\in\Gamma}(1 + |x|^m) \left| \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \varphi(x_1, \ldots, x_n) \right| < \infty,$$

where $m = 0, 1, 2, \ldots$, $k = (k_1, \ldots, k_n)$, $|k| = k_1 + \cdots + k_n$. The value of $f(u) \in S'(\Gamma)$ at $\varphi(u) \in S(\Gamma)$ is $(f(u), \varphi(u))$. The Laplace transform

$$\widetilde{f}(z) = (f(u), e^{i(z,u)}), \qquad z = x + iy \in T^C = \mathbf{R}^n + iC,$$

realises an isomorphism between the convolutional algebra $S'(\Gamma)$ and the algebra $H(C)$ of the functions which are holomorphic in T^C and obey, for some M, a, b, the bound

$$|\widetilde{f}(z)| \leq M(1 + |z|)^a / [\Delta_C(y)]^b,$$

$\Delta_C(y)$ is the distance between y and the boundary of the cone C (for more details, see (Vladimirov *et al.*, 1988)).

Let $K_C(z)$ stand for the Cauchy kernel of the tubular domain T^C, that is,

$$K_C(z) = \int_\Gamma e^{i(z,u)} du, \qquad z \in T^C.$$

DEFINITION 1.7.6. A cone Γ is said to be *regular* if $1/K_C(z)$ belongs to the algebra $H(C)$.

Tauberian theorems 1.7.2, 1.7.3, 1.7.5, and Abelian theorem 1.7.6 remain true for arbitrary homogeneous regular cones.

For $n \leq 3$, all closed convex acute solid cones are regular. For $n \geq 4$, there exist non-regular closed convex acute solid cones (Danilov, 1985).

All n-faced cones are regular. If a cone Γ is self-dual ($\Gamma^* = \Gamma$) and homogeneous, then it is regular. In particular, the future light cone

$$V_+ = \{x: x = (x_0, \widetilde{x}),\ \widetilde{x} = (x_1, \ldots, x_n),\ x_0 \geq |\widetilde{x}|\}$$

is regular (for more details, see (Vladimirov *et al.*, 1988)).

Let us formulate (in a slightly shortened form) a one-dimensional Tauberian theorem of Drozhzhinov–Zavyalov type for asymptotic expansions of Laplace transforms (Drozhzhinov, Zavyalov, 1995b).

Let μ be a finite non-negative measure on $\mathbf{R}^1_+$ whose Laplace transform is $\widetilde{\mu}(y)$:

$$\widetilde{\mu}(y) = \int_0^\infty e^{-yu}\mu(du), \qquad y \geq 0.$$

THEOREM 1.7.7. *Let $r(t)$ be a regularly varying function with index $-\alpha$, where $n < \alpha < n+1$ for some $n = 0, 1, \dots$ Then the following assertions are equivalent:*

(1) *There exist constants $c_0, \dots, c_n$ such that*

$$\widetilde{\mu}(y) = \sum_{j=0}^{n} c_j y^j + h(y), \qquad y > 0,$$

and

$$\frac{h(y)}{r(1/y)} \to A, \qquad y \downarrow 0;$$

(2) *for any integer $l \geq 0$, as $t \to \infty$*

$$\int_0^t u^l \mu(du) = (-1)^l l!\, c_l + t^l r(t)\left[\frac{A}{(\alpha + l)\Gamma(\alpha)} + o(1)\right],$$

for $l > n$ we can set $c_l = 0$;

(3) *for some integer $m \geq 0$ there exists a constant c_m such that, as $t \to \infty$,*

$$\int_0^t u^m \mu(du) = (-1)^m m!\, c_m + t^m r(t)\left[\frac{A}{(\alpha + m)\Gamma(\alpha)} + o(1)\right];$$

(4) *as $t \to \infty$,*

$$\mu((t, \infty)) = r(t)\left[\frac{A}{\alpha\Gamma(\alpha)} + o(1)\right].$$

REMARK 1.7.1. The implication $1 \Rightarrow 4$ is proved in (Nevels, 1974), see also Problem 15 in Chapter XVII of (Feller, 1966).

REMARK 1.7.2. In (Drozhzhinov, Zavyalov, 1995a), a Tauberian theorem for asymptotic expansions of Laplace transforms of measures concentrated in the positive octant is also proved.

Let us present a Tauberian theorem of Drozhzhinov–Zavyalov type for characteristic functions (Drozhzhinov, Zavyalov, 1995b)). Let $f(x)$ be a characteristic function of some random variable with distribution function $F(x)$:

$$f(x) = \int_{-\infty}^{+\infty} e^{itx}\, dF(t), \qquad x \in \mathbf{R}^1.$$

For a given $\varepsilon > 0$, we set

$$\Phi_+(y) = \frac{1}{2\pi i}\int_{-\varepsilon}^{+\varepsilon} \frac{f(u)\,du}{u - iy}, \qquad \Phi_-(y) = \frac{1}{2\pi i}\int_{-\varepsilon}^{+\varepsilon} \frac{f(u)\,du}{u + iy}.$$

THEOREM 1.7.8. *Let $r(t)$ be a regularly varying function with index $-\alpha$, where $n < \alpha < n+1$ for some $n = 0, 1, \ldots$ If there exist constants $c_0, c_1, \ldots, c_n$ such that*

$$\Phi_+(y) = \sum_{j=0}^{n} c_j y^j + h(y), \qquad y > 0,$$

and

$$h(y) \sim r(1/y), \qquad y \to 0_+,$$

then

$$1 - F(t) \sim \frac{1}{\alpha\Gamma(\alpha)} r(t), \qquad t \to +\infty.$$

Similarly, if

$$\Phi_-(y) = \sum_{j=0}^{n} c_j y^j + h(y), \qquad y > 0,$$

and

$$h(y) \sim r(1/y), \qquad y \to 0_+,$$

then

$$F(-t) \sim \frac{1}{\alpha\Gamma(\alpha)} r(t), \qquad t \to +\infty.$$

1.8. Three multidimensional Tauberian theorems

In this section we give proofs of three multidimensional Tauberian theorems of Drozhzhinov–Zavyalov type (Drozhzhinov, Zavyalov, 1984; Drozhzhinov, Zavyalov, 1986b; Vladimirov *et al.*, 1988) obtained by the author of this book. These theorems will find application in Chapter 4 where the asymptotic behaviour of infinitely divisible distributions in a cone will be studied. The method to prove them combines those used in (Drozhzhinov, Zavyalov, 1986b) and (Yakymiv, 1982). Let Γ be a closed convex acute solid cone in $\mathbf{R}^n$. As before, we set $G = \operatorname{int}\Gamma$, $C = \operatorname{int}\Gamma^*$.

Let $U = \{U_k,\ k \in I \subseteq [0,\infty)\}$ stand for an arbitrary family of linear operators in $\mathbf{R}^n$ which leave invariant the cone Γ:

$$U_k\Gamma = \Gamma \quad \forall k \in I, \quad J_k = \det U_k. \tag{1.8.1}$$

We assume that ∞ is a limit point of the set I. The operator $V_k = (U_k^*)^{-1}$ leaves invariant the dual cone Γ^*. For an arbitrary operator U_k, we set

$$\Lambda(k) = \sup_{|e|=1} |U_k e|, \qquad \lambda(k) = \inf_{|e|=1} |U_k e|. \tag{1.8.2}$$

To the operator V_k, we put into correspondence

$$\frac{1}{\lambda(k)} = \sup_{|e|=1} |V_k e|, \qquad \frac{1}{\Lambda(k)} = \inf_{|e|=1} |V_k e|.$$

Let us recall a definition from the preceding section.

DEFINITION 1.8.1. We say that a family $U = \{U_k,\ k \in I\}$ is a family of first type if $\Lambda(k) \to \infty$ and $\lambda(k) \to \infty$ as $k \to \infty$, $k \in I$; of second type if there exists $b > 0$ such that $\Lambda(k) \to \infty$ as $k \to \infty$, $k \in I$, and $\lambda(k) \geq b > 0$, $k \in I$; of third type if $\Lambda(k) \to \infty$ as $k \to \infty$, $k \in I$.

We introduce functions which regularly vary in G along the family U.

DEFINITION 1.8.2. We say that a function $f(x)$, which is defined, positive, and measurable in G, is regularly varying in G along a family $U = \{U_k,\ k \in I\}$ and write $f \in R(U, G)$, if for some vector $e \in G$ and all $x \in G$ as $x_k \to x$, $k \to \infty$, $k \in I$,

$$\frac{f(U_k x_k)}{f(U_k e)} \to \varphi(x) > 0, \qquad \varphi(x) < \infty. \tag{1.8.3}$$

From (1.8.1) and Theorem 1.4.2 it follows that

$$\frac{f(U_k x)}{f(U_k e)} \overset{x\in K}{\rightrightarrows} \varphi(x) \in (0, \infty), \quad k \to \infty, \quad k \in I, \tag{1.8.4}$$

for any compact $K \subset G$, while $\varphi(x)$ is continuous in G. In accordance with (1.8.3) we write $\varphi = H_e(U, G)$.

REMARK 1.8.1. The set $R(U, G)$ does not depend on the vector $e \in G$. Further, if (1.8.3) holds for some vector $e \in G$, then it holds for all vectors $e_1 \in G$, and the function φ is multiplied by

$$\lim_{k\to\infty,\ k\in I} \frac{f(U_k e)}{f(U_k e_1)}.$$

DEFINITION 1.8.3. A function $f(x)$ defined in G is said to be completely admissible for a family of operators $U = \{U_k, k \in I\}$, if $f \in R(U, G)$, f is locally summable in G and there exists k_0 such that

$$\frac{f(U_k x)}{f(U_k e)} \leq \eta(x), \quad k > k_0, \quad k \in I, \quad x \in G, \tag{1.8.5}$$

where e is a fixed vector in G and η is a tempered function in G:

$$\int_G \frac{\eta(x)}{1 + |x|^q} < \infty$$

for some q.

REMARK 1.8.2. Definition 1.8.3 is equivalent to the definition of a 0-completely admissible function for a family $U = \{U_k,\ k \in I\}$ in (Vladimirov *et al.*, 1988, Chapter II, Section 5.2).

DEFINITION 1.8.4. A function $f(x)$ defined in G, is said to be *admissible for a cone* Γ, if for an arbitrary family of linear operators $U = \{U_k,\ k \in I\}$ which leave invariant the cone Γ there exists a subfamily $V = \{U_k,\ k \in J \subseteq I\}$ (J is unbounded) for $f(x)$ is completely admissible.

Let $\mathbb{D}(\Gamma)$ denote the set of all admissible for the cone Γ functions.

DEFINITION 1.8.5. A function $f(x)$ defined in G is said to be admissible of type m, $m = 1, 2, 3$, for a cone Γ if for an arbitrary family of linear operators of type m $U = \{U_k,\ k \in I\}$ which leave invariant the cone Γ there exists a subfamily $V = \{U_k,\ k \in J \subseteq I\}$ (J is unbounded) for $f(x)$ is completely admissible.

Let $\mathbb{D}_m(\Gamma)$, $m = 1, 2, 3$, denote the set of all admissible functions of type m for a cone Γ.

Let $\mathbb{R}_m(\Gamma)$, $m = 1, 2, 3$, denote the set of all functions $f(x)$ defined in G such that for an arbitrary family of linear operators of type m $U = \{U_k,\ k \in I\}$ which leave invariant the cone Γ there exists a subfamily $V = \{U_k,\ k \in J \subseteq I\}$ (J is unbounded) such that $f \in R(V, \Gamma)$ (see Definition 1.8.2).

Let $\mathbb{R}(\Gamma)$ denote the set of all functions $f(x)$ defined in G such that for any family of linear operators $U = \{U_k,\ k \in I\}$ which leave invariant the cone Γ there exists a subfamily $V = \{U_k,\ k \in J \subseteq I\}$ (J is unbounded) such that $f \in R(V, \Gamma)$ (see Definition 1.8.2).

It is clear that

$$\mathbb{D}(\Gamma) \subseteq \mathbb{D}_3(\Gamma) \subseteq \mathbb{D}_2(\Gamma) \subseteq \mathbb{D}_1(\Gamma)$$

and

$$\mathbb{R}(\Gamma) \subseteq \mathbb{R}_3(\Gamma) \subseteq \mathbb{R}_2(\Gamma) \subseteq \mathbb{R}_1(\Gamma).$$

Let us give examples of functions of $\mathbb{R}(\Gamma)$ and $\mathbb{D}(\Gamma)$.

Let

$$\mathbf{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n,\ x_i \geq 0\ \forall i = 1, \dots, n\}$$

be the positive coordinate octant,

$$V_n^+ = \left\{x = (x_0, x_1, \dots, x_n) \in \mathbf{R}^{n+1},\ x_0 \geq \sqrt{x_1^2 + \cdots + x_n^2}\right\}$$

be the future light cone in $\mathbf{R}^{n+1}$.

EXAMPLE 1.8.1. Let $f(x)$ be a positive continuously differentiable function in $\operatorname{int} \mathbf{R}^n_+$ which satisfies the relation

$$a \leq \frac{x_j \frac{\partial}{\partial x_j} f(x)}{f(x)} \leq b, \quad x_j > 0, \quad j = 1, \dots, n,$$

for some $a, b \in (-\infty, \infty)$. Then (see (Vladimirov *et al.*, 1988, Chapter II, Section 5.3, Theorem 1)) $f \in \mathbb{R}(\mathbf{R}^n_+)$. In particular, if $a > -1$, then $f \in \mathbb{D}(\mathbf{R}^n_+)$.

EXAMPLE 1.8.2. Let $f(x)$ be a positive continuously differentiable function in $\operatorname{int} V_n^+$, and

$$a \leq \frac{(l, x)(l, \nabla f(x))}{f(x)} \leq b, \quad x \in V_n^+, \quad l \in \partial V_n^+, \quad |l| = 1,$$

for some $a, b \in (-\infty, \infty)$, where $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. Then (see (Vladimirov *et al.*, 1988, Chapter II, Section 5.3, Theorem 2)) $f \in \mathbb{R}(V_+^n)$. In particular, if $a > -1$, then $f \in \mathbb{D}(V_n^+)$.

Let Γ be a closed convex acute solid cone in $\mathbf{R}^n$ with apex at zero. We consider a function χ which admits the representation

$$\chi(t) = \eta(t)\varphi(t), \qquad t \in (0, \infty), \tag{1.8.6}$$

where the functions η and φ satisfy the following conditions:

(1) there exist positive c_1 and c_2 such that

$$c_1 \le \eta(t) \le c_2, \qquad t \in (0, \infty); \tag{1.8.7}$$

(2) the relation

$$\omega(\lambda, t) = \frac{\eta(\lambda t)}{\eta(t)} \xrightarrow[]{t\in(0,\infty)} 1, \qquad \lambda > 1, \tag{1.8.8}$$

holds true;

(3) The function φ is positive, continuously differentiable, and satisfies the inequalities

$$a \le \frac{t\varphi'(t)}{\varphi(t)} \le b, \qquad 0 < t < \infty, \tag{1.8.9}$$

for some $a, b \in (-\infty, \infty)$.

EXAMPLE 1.8.3. Let the function χ satisfy (1.8.6)–(1.8.9), and let the vectors $l_i \in \Gamma^*$, numbers $A_i > 0$, and $\alpha \in \mathbf{R}^1$ be fixed. Then (see (Vladimirov *et al.*, 1988, Chapter II, Section 5.4)) $f(x) = \chi(\omega(x)) \in \mathbb{R}(\Gamma)$, where

$$\omega(x) = \frac{\sum_{i=1}^{p} A_i (l_i, x)^{\alpha_i}}{\sum_{i=p+1}^{p+q} A_i (l_i, x)^{\alpha_i}}, \qquad x \in \Gamma.$$

In particular, if $-\alpha_i a < 1$, $i = 1, \dots, p$, and $\alpha_i b < 1$, $i = p+1, \dots, p+q$, then $f(x) \in \mathbb{D}(\Gamma)$.

Let $M_1^+(G)$ ($M_2^+(G)$, respectively) denote the set of all non-decreasing (non-increasing) functions in G which are upper continuous, that is, such that $f(x) = f(x_+)$ for all $x \in G$, and let $M^+(G) = M_1^+(G) \cup M_2^+(G)$ (see the beginning of Section 1.2).

As before, let $\widehat{f}(y)$ and $\widetilde{\mu}(y)$ denote, respectively, the Laplace transform of a function f and of a measure μ on Γ:

$$\widehat{f}(y) = \int_\Gamma e^{-(y,x)} f(x)\, dx \qquad \widetilde{\mu}(y) = \int_\Gamma e^{-(y,x)} \mu(dx),$$

provided that there exist for $y \overset{T}{>} a$ with some $a \in T = \Gamma^*$.

Let Γ be an arbitrary closed convex acute solid cone which admits a family of linear transformations $U = \{U_k,\ k \in I\}$ which leave invariant the cone Γ (see (1.8.1)). The following Tauberian theorem is true.

THEOREM 1.8.1. *Let* $f(x) = u(x)v(x)$, $x \in G$, *where* $u(x) \in R(U, G)$ *(see Definition* 1.8.2*)*, $v(x) \in M^+(G)$, $v(x) \geq 0$, $x \in G$, $\rho_k > 0$, $k \in I$, *and let for all* $y \in C$ *there exist* $\widehat{f}(y)$. *We assume that for all* $y \in C$

$$\frac{\widehat{f}(V_k y)}{J_k \rho_k} \to \psi(y) < \infty, \quad k \to \infty, \quad k \in I, \quad V_k = (U_k^*)^{-1}, \quad J_k = \det U_k. \tag{1.8.10}$$

Then there exists a function $\varphi(x) < \infty$, $x \in G$, *such that*

$$\frac{f(U_k x)}{\rho_k} \to \varphi(x), \quad k \to \infty, \quad k \in I, \tag{1.8.11}$$

almost everywhere in G *and a measure* μ *on* Γ *such that* φ *is its density in* G *and*

$$\psi(y) = \widetilde{\mu}(y) < \infty \qquad \forall y \in C. \tag{1.8.12}$$

In particular, if $\mu(\partial\Gamma) = 0$, *then*

$$\psi(y) = \widehat{\varphi}(y) < \infty, \qquad \forall y \in C. \tag{1.8.13}$$

If the function $\varphi(x)$ *is continuous in* G, *then relation* (1.8.11) *holds uniformly in* $x \in K$ *for any compact* $K \subset G$.

REMARK 1.8.3. Theorem 1.8.1 is an analogue of Theorem 1 in (Vladimirov *et al.*, 1988, Chapter II, Section 4.3). The difference of our theorem consists of weakening the Tauberian condition and breaking the regularity of the cone Γ.

PROOF OF THEOREM 1.8.1. Let $\mathfrak{A}$ be the set of all bounded Borel sets in $\mathbf{R}^n$. In what follows, the measures are non-negative, σ-finite, and concentrated on Γ. We define a family of measures $\{\mu_k,\ k \in I\}$ on $\mathfrak{A}$ by the formula

$$\mu_k(A) = \int_{A \cap \Gamma} \frac{f(U_k x)}{\rho_k} dx, \qquad A \in \mathfrak{A}. \tag{1.8.14}$$

Since

$$\widetilde{\mu}_k(y) = \frac{\widehat{f}(V_k(y))}{J_k \rho_k}, \qquad y \in C, \tag{1.8.15}$$

from (1.8.10) it follows that for any $y \in C$

$$\widetilde{\mu}_k(y) \to \psi(y) < \infty, \quad k \to \infty, \quad k \in I. \tag{1.8.16}$$

By virtue of Theorem 1.3.2, there exists a measure μ on Γ such that

$$\mu_k \Rightarrow \mu, \quad k \to \infty, \quad k \in I, \tag{1.8.17}$$

and relation (1.8.12) holds true. We fix an arbitrary vector $e \in G$ and set for $x \in G$

$$u_k(x) = \frac{u(U_k x)}{u(U_k e)}, \qquad v_k(x) = \frac{v(U_k x)}{\rho_k} u(U_k e). \tag{1.8.18}$$

For definiteness, let $v \in M_2^+(G)$. We observe that

$$v(Ux) \in M_2^+(G) \tag{1.8.19}$$

for any linear operator U which leaves invariant the cone Γ, that is, $U\Gamma = \Gamma$. It is easily seen, indeed, that since $U\Gamma = \Gamma$, we obtain $U(\partial\Gamma) = \partial\Gamma$ and $U(G) = G$. Therefore, for $x, y \in G$ and $x \overset{\Gamma}{<} y$ we find that $Ux, Uy, U(y-x) \in G$, because $y - x \in G$. Then $Ux \overset{\Gamma}{<} Uy$, hence we obtain $v(Ux) \geq v(Uy)$. Next, by virtue of Lemma 1.2.2,

$$\lim_{y \to x,\ y \overset{\Gamma}{>} x} v(Uy) = \lim_{Uy \to Ux,\ Uy \overset{\Gamma}{>} Ux} v(Uy) = v((Ux)_+) = v(Ux)$$

(the last equality is true by the definition of $M_2^+(G)$). In other words, (1.8.19) holds. By virtue of (1.8.19), $v_k(x) \in M_2^+(G)$ for any $r \in I$. Let us demonstrate that for any $x \in G$ there exist $k_x \in I$ and $c_x < \infty$ such that for $k > k_x$, $k \in I$

$$v_k(x) \leq c_x. \tag{1.8.20}$$

It is easy to see that for any closed ball A with centre at x such that $A \subset G$ and $\mu(\partial A) = 0$, by virtue of (1.8.17) and (1.8.15),

$$\mu(A) = \lim_{k \to \infty,\ k \in I} \int_A u_k(y) v_k(y)\, dy;$$

so there exists $k_{1x} \in I$ such that for all $k > k_{1x}$, $k \in I$

$$\int_A u_k(y) v_k(y)\, dy < \mu(A) + 1. \tag{1.8.21}$$

But

$$\int_A u_k(y) v_k(y)\, dy \geq \inf_{y \in A} u_k(y) \int_B v_k(y)\, dy \geq \inf_{y \in A} u_k(y) |B| v_k(x), \tag{1.8.22}$$

where $B = \{y \colon y \in A,\ y \overset{\Gamma}{<} x\}$ in the case where v does not increase and $B = \{y \colon y \in A,\ y \overset{\Gamma}{>} x\}$ in the case where v does not decrease. Let $H_e(U, G) = \varphi_1$. Since $u \in R(U, G)$, we obtain

$$u_k(y) \overset{y \in A}{\Longrightarrow} \varphi_1(y) > 0, \quad k \to \infty, \quad k \in I.$$

Therefore, there exists $k_x > k_{1x}$, $k_x \in I$, such that for $k > k_x$, $k \in I$,

$$\inf_{y \in A} u_k(y) \geq b > 0. \tag{1.8.23}$$

Inequality (1.8.20) with $c_x = (\mu(A) + 1)/(|B|b)$ now follows from (1.8.21)–(1.8.23). By virtue of Theorem 1.2.1, the set of functions $\{v_k(x),\ k > k_x,\ k \in I\}$ is weakly precompact. For some unbounded set $J \subseteq I$, let

$$v_k \Rightarrow h, \quad k \to \infty, \quad k \in J. \tag{1.8.24}$$

Let us demonstrate that the function $\varphi_1(x)h(x)$ is the density of the measure μ with respect to the Lebesgue measure in G. Let a set $A \in \mathfrak{A}$, $\overline{A} = A \cup \partial A \subset G$. Then there exist $\varepsilon > 0$ and $\lambda > 0$ such that for any $x \in A$ we obtain $\varepsilon e \overset{\Gamma}{<} x \overset{\Gamma}{<} \lambda e$. Since $u \in R(U, G)$, there exist k_0 and $c < \infty$ such that for $k > k_0$, $k \in I$, and $x \in \overline{A}$

$$\frac{u(U_k x)}{u(U_k e)} \leq c. \tag{1.8.25}$$

From (1.8.20) and (1.8.25), for $k > \max(k_0, k_{\varepsilon e}, k_{\lambda e})$, $k \in I$ and $x \in A$ we see that

$$\frac{f(U_k x)}{\rho_k} = \frac{u(U_k x)}{u(U_k e)} v_k(x) \leq c \max(c_{\varepsilon e}, c_{\lambda e}). \tag{1.8.26}$$

By virtue of the Lebesgue theorem, in view of (1.8.24) and (1.8.26), we obtain

$$\int_A \frac{f(U_k x)}{\rho_k} dx \to \int_A \varphi_1(x) h(x)\, dx, \quad k \to \infty, \quad k \in J. \tag{1.8.27}$$

We assume that $\mu(\partial A) = 0$. Then, along with (1.8.27), the relation

$$\mu_k(A) = \int_A \frac{f(U_k x)}{\rho_k} dx \to \mu(A), \quad k \to \infty, \quad k \in I, \tag{1.8.28}$$

holds (see (1.8.17)). From (1.8.27) and (1.8.28) it follows that

$$\int_A \varphi_1(x) h(x)\, dx = \mu(A). \tag{1.8.29}$$

By the last inequality, the function h is uniquely defined by μ up to a set of Lebesgue measure zero. But $h \in M^+(G)$ in view of (1.8.24); hence

$$h(x) = \lim_{y \downarrow x} h(y)$$

for any $x \in G$, so h is uniquely defined by μ. Since the limit function in (1.8.24) is unique, we conclude that

$$v_k \Rightarrow h, \quad k \to \infty, \quad k \in I. \tag{1.8.30}$$

Thus, in every point x of continuity of the function h we see that

$$\frac{f(U_k x)}{\rho_k} = \frac{u(U_k x)}{u(U_k e)} v_k(x) \to \varphi_1(x) h(x), \quad k \to \infty, \quad k \in I, \tag{1.8.31}$$

and relation (1.8.29) holds. From (1.8.31) it follows that relation (1.8.11) with $\varphi(x) = \varphi_1(x)h(x)$ is true. From (1.8.29) it follows that $\varphi = \varphi_1 h$ is the density of the measure μ with respect to the Lebesgue measure in G. Finally, (1.8.12) follows from relations (1.8.16) and (1.8.17).

Now let $\varphi(x)$ be continuous in G. It remains to show that relation (1.8.11) holds uniformly in $x \in K$ for any compact $K \subset G$. By virtue of Theorem 1.4.2, it suffices to verify that

$$\frac{f(U_k x_k)}{\rho_k} \to \varphi(x) \tag{1.8.32}$$

for any $x \in G$ provided that $x_k \in G$, $x_k \to x$ as $k \to \infty$, $k \in I$. Since $\varphi(x)$ is continuous in G and $\varphi_1(x)$ is positive and continuous in G, the function $h(x)$ is continuous in G. We fix some $a \in (0, 1)$, $b \in (1, \infty)$. For definiteness, let $h \in M_2^+(G)$. Then the inequalities

$$\limsup_{k\to\infty,\, k\in I} \frac{f(U_k x_k)}{\rho_k} = \limsup_{k\to\infty,\, k\in I} \frac{u(U_k x_k)}{u(U_k e)} v_k(x_k) \le \lim_{k\to\infty,\, k\in I} \frac{u(U_k x_k)}{u(U_k e)} v_k(ax) = \varphi_1(x)h(ax) \tag{1.8.33}$$

are true (the last one follows from (1.8.3) and (1.8.30) in view of the fact that h is continuous in ax). Passing in (1.8.33) to the limit as $a \uparrow 1$ and recalling that h is continuous at the point x, we obtain

$$\limsup_{k\to\infty,\, k\in I} \frac{f(U_k x_k)}{\rho_k} \le \varphi_1(x)h(x). \tag{1.8.34}$$

Further, the lower bound

$$\liminf_{k\to\infty,\, k\in I} \frac{f(U_k x_k)}{\rho_k} = \liminf_{k\to\infty,\, k\in I} \frac{u(U_k x_k)}{u(U_k e)} v_k(x_k) \ge \lim_{k\to\infty,\, k\in I} \frac{u(U_k x_k)}{u(U_k e)} v_k(bx) = \varphi_1(x)h(bx) \tag{1.8.35}$$

is true. By passing to the limit as $b \downarrow 1$ in (1.8.35), we obtain

$$\liminf_{k\to\infty,\, k\in I} \frac{f(U_k x_k)}{\rho_k} \ge \varphi_1(x)h(x). \tag{1.8.36}$$

Now (1.8.32) with $\varphi(x) = \varphi_1(x)h(x)$ follows from (1.8.34) and (1.8.36). In the case where $h \in M_1^+(G)$, (1.8.32) is proved similarly: the inequalities are replaced by the opposite ones, and lim sup, lim inf are swapped. The proof of the theorem is thus complete. □

As before, let Γ be an arbitrary closed convex acute solid cone which admits a family of linear transformations $U = \{U_k,\ k \in I\}$ which leave Γ invariant. The following Tauberian comparison theorem is true.

THEOREM 1.8.2. *Let $f(x) = u(x)v(x)$, $x \in G$), where $u(x) \in R(U, G)$ (see Definition* 1.8.2*), $v(x) \in M^+(G)$, $v(x) \ge 0$, $x \in G$, and for all $y \in C$ let there exist $\hat{f}(y)$. We assume that the function $g(x)$ is completely admissible for a family $U = \{U_k,\ k \in I\}$ (see Definition* 1.8.3*). If*

$$\frac{\hat{f}(V_k y)}{\hat{g}(V_k y)} \to 1, \quad k \to \infty, \quad k \in I, \tag{1.8.37}$$

for any $y \in C$, $V_k = (U_k^*)^{-1}$, then the limit

$$\frac{f(U_k x)}{g(U_k x)} \xrightarrow{x\in K} 1, \quad k \to \infty, \quad k \in I, \tag{1.8.38}$$

exists for any compact $K \subset G$.

PROOF. In view of complete admissibility of the function g for the family $U = \{U_k, k \in I\}$, for some fixed vector $e \in G$ we see that

$$\frac{g(U_k x)}{g(U_k e)} \overset{x \in K}{\Longrightarrow} \varphi(x) > 0, \quad k \to \infty, \quad k \in I, \tag{1.8.39}$$

for any compact $K \subset G$. We set $\rho_k = g(U_k e)$, $k \in I$, and demonstrate that

$$\frac{f(U_k x)}{\rho_k} \overset{x \in K}{\Longrightarrow} \varphi(x), \quad k \to \infty, \quad k \in I, \tag{1.8.40}$$

for any compact $K \subset G$. It is easily seen indeed that for $y \in C$

$$\frac{\widehat{f}(V_k y)}{J_k \rho_k} = \frac{\widehat{f}(V_k y)}{\widehat{g}(V_k y)} \frac{\widehat{g}(V_k y)}{J_k g(U_k e)}, \tag{1.8.41}$$

and by virtue of the Lebesgue theorem, in view of (1.8.39) and bound (1.8.5) for the function $g(x)$, we obtain, as $k \to \infty$, $k \in I$,

$$\frac{\widehat{g}(V_k y)}{J_k g(U_k e)} = \int_G \frac{g(U_k x)}{g(U_k e)} e^{-(y,x)} dx \to \widehat{\varphi}(y) < \infty$$

for any $y \in C$. So from (1.8.37) and (1.8.41) it follows that

$$\frac{\widehat{f}(V_k y)}{J_k \rho_k} \to \widehat{\varphi}(y) < \infty, \quad k \to \infty, \quad k \in I, \quad \forall y \in C.$$

Thus, all hypotheses of Theorem 1.8.1 with $\psi(y) = \widehat{\varphi}(y)$ are satisfied. Since the function $\varphi(x)$ is continuous in G, making use of Theorem 1.8.1 we conclude that relation (1.8.40) holds true, and

$$\frac{f(U_k x)}{g(U_k x)} = \frac{f(U_k x)}{\rho_k} \left(\frac{g(U_k x)}{g(U_k e)} \right)^{-1}. \tag{1.8.42}$$

Relation (1.8.38) follows from relations (1.8.39), (1.8.40) and (1.8.42) in view of positivity of $\varphi(x)$. The theorem is proved. □

Let $\Delta_\Gamma(x)$ be the distance between the point $x \in \Gamma$ and the boundary of the cone Γ, and let $\Delta_T(x)$ be the distance between the point $x \in T$ and the boundary of the cone $T = \Gamma^*$.

DEFINITION 1.8.6. A convex cone Γ is said to be *homogeneous* if for arbitrary points x_1 and x_2 in $G = \operatorname{int} \Gamma$ there exists a non-singular linear transformation U (generally speaking, depending on x_1 and x_2) which leaves invariant the cone Γ such that $Ux_1 = x_2$.

Let Γ be an arbitrary closed convex acute solid homogeneous cone in $\mathbf{R}^n$ with apex at zero. The following Tauberian theorem is true.

THEOREM 1.8.3. *Let* $f(x) = u(x)v(x)$, $x \in G$, $v(x) \in M^+(G)$, $v(x) \geq 0$, *and for all* $y \in C$ *let there exist* $\widehat{f}(y)$. *The following assertions are true.*

(1) *If* $g(x) \in \mathbb{D}_1(\Gamma)$ *(see Definition* 1.8.5*),* $u(x) \in \mathbb{R}_1(\Gamma)$*, and the limit*

$$\frac{\widehat{f}(y)}{\widehat{g}(y)} \to 1 \tag{1.8.43}$$

exists as $y \to 0$*,* $y \in C$*, then*

$$\frac{f(x)}{g(x)} \to 1, x \in G, \quad \Delta_\Gamma(x) \to \infty. \tag{1.8.44}$$

(2) *If* $g(x) \in \mathbb{D}_2(\Gamma)$ *(see Definition* 1.8.5*),* $u(x) \in \mathbb{R}_2(\Gamma)$*, and for any* $b > 0$ *the limit*

$$\frac{\widehat{f}(y)}{\widehat{g}(y)} \to 1, \quad \Delta_T(y) \to 0, \quad y \in C, \quad |y| < b, \tag{1.8.45}$$

exists, then for any $\delta > 0$

$$\frac{f(x)}{g(x)} \to 1, \quad x \in G, \quad \Delta_\Gamma(x) \geq \delta. \tag{1.8.46}$$

(3) *If* $g(x) \in \mathbb{D}_3(\Gamma)$ *(see Definition* 1.8.5*),* $u(x) \in \mathbb{R}_3(\Gamma)$*, and the limit*

$$\frac{\widehat{f}(y)}{\widehat{g}(y)} \to 1, \quad \Delta_T(y) \to 0, \quad y \in C, \tag{1.8.47}$$

exists, then

$$\frac{f(x)}{g(x)} \to 1, \quad x \in G, \quad |x| \to \infty. \tag{1.8.48}$$

PROOF. We begin with proving the first assertion. We assume the contrary, that is, let (1.8.43) be true but (1.8.44) be broken. Then there exists $\varepsilon > 0$ and a sequence $x_k \in G$, $\Delta_\Gamma(x_k) \geq k$, $k \in \mathbf{N}$, such that

$$\left| \frac{f(x_k)}{g(x_k)} - 1 \right| > \varepsilon, \qquad k \in \mathbf{N}. \tag{1.8.49}$$

We fix $a \in G$, $|a| = 1$, and consider a family of linear operators $U = \{U_k,\ k \in \mathbf{N}\}$, $U_k\Gamma = \Gamma$, such that $U_k a = x_k$. Such operators exist because of the homogeneity of Γ. Since $|U_k a| = |x_k| \geq k$, we see that

$$k \leq \Delta_\Gamma(x_k) = \Delta_\Gamma(U_k a) = \frac{\lambda(k)}{p(a)},$$

where $\lambda(k)$ are defined by relation (1.8.2) and $p(a) > 0$ (see Lemma 2 in (Vladimirov *et al.*, 1988, Chapter II, Section 5.1)). Then $\lambda(k) \to \infty$ as $k \to \infty$, $k \in \mathbf{N}$, and therefore, $\Lambda(k) \to \infty$ as $k \to \infty$, $k \in \mathbf{N}$, too (see (1.8.2)). Thus, the family $\{U_k, k \in \mathbf{N}\}$ is a family of first type, and since $g(x) \in \mathbb{D}_1(\Gamma)$, there exists a subfamily $\{U_k,\ k \in L \subseteq \mathbf{N}\}$ (L is unbounded) for which $g(x)$ appears to be completely admissible. By the definition of $\mathbb{R}_1(\Gamma)$, in the family $\{U_k,\ k \in L\}$ a subfamily $U = \{U_k,\ k \in M \subseteq L\}$ exists (M is unbounded) such that $u \in R(U, \Gamma)$. It is clear that the function $g(x)$ remains completely admissible for the family U. For an arbitrary fixed vector $y \in C$ and $k \in \mathbf{N}$, we see that $V_k y \in C$ and

$$|V_k y| \leq |y| \sup_{|e|=1} |V_k e| = \frac{|y|}{\lambda(k)} \to 0, \quad k \to \infty, \quad k \in \mathbf{N}.$$

Therefore, by condition (1.8.43)

$$\frac{\widehat{f}(V_k y)}{\widehat{g}(V_k y)} \to 1, \quad k \to \infty, \quad k \in M. \tag{1.8.50}$$

Thus, all hypotheses of Theorem 1.8.2 are satisfied. By virtue of this theorem,

$$\frac{f(U_k a)}{g(U_k a)} = \frac{f(x_k)}{g(x_k)} \to 1, \quad k \to \infty, \quad k \in M.$$

The last relation contradicts inequality (1.8.49) for $k \in M$. Assertion 1 of the theorem is thus proved. The proofs of assertions 2 and 3 repeat the above with account for the fact that the families of operators $\{U_k,\ k \in \mathbf{N}\}$ are of second and third types, respectively (see (Vladimirov *et al.*, 1988, Chapter II, Section 6.2, Theorem 2)). □

1.9. Remarks to Chapter 1

The regularly varying functions of one variable were first introduced by J. Karamata in (Karamata, 1930a) who proved Tauberian theorems for them (Karamata, 1930b; Karamata, 1931a; Karamata, 1931b). Multidimensional extensions of regularly varying functions are studied in (Bajšanski, Karamata, 1969; Rvacheva, 1962; Resnick, 1986; Resnick, 1987; Resnick, 1991; Greenwood, Resnick, 1979; de Haan, 1985; Omey, 1989; de Haan *et al.*, 1984; Klüppelberg *et al.*, 2003; Meerschaert, 1986; Meerschaert, 1988; Meerschaert, 1993; Meerschaert, Scheffler, 1999; Ostrogorski, 1995; Ostrogorski, 1997a; Ostrogorski, 1997b; Ostrogorski, 1988; Stam, 1977; Drozhzhinov, Zavyalov, 1984; Drozhzhinov, Zavyalov, 1986a; Drozhzhinov, Zavyalov, 1986b; Drozhzhinov, Zavyalov, 1990; Drozhzhinov, Zavyalov, 1992; Drozhzhinov, Zavyalov, 1995a; Zaigraev, Nagaev, 2003; Kozlov, 1983; Molchanov, 1993; Yakymiv, 1982).

In (Bajšanski, Karamata, 1969), continuous functions $f\colon G \to \mathbf{R}_+$ are considered, where G is an arbitrary topological group where a filter $\mathfrak{U}$ of open convex sets in G with countable base is given. The filter $\mathfrak{U}$ is thought of as G-invariant, that is, $Uh \in \mathfrak{U}$ and $hU \in \mathfrak{U}$ for any set $U \in \mathfrak{U}$ and any element $h \in U$. According to (Bajšanski, Karamata, 1969), a function f is said to be regularly varying with respect to filter $\mathfrak{U}$ if the limit

$$\lim_{g \to \infty} \frac{f(gh)}{f(g)} = \varphi(h)$$

exists for any $h \in G$, where $g \to \infty$ means convergence with respect to the filter. In (Bajšanski, Karamata, 1969), a theorem about uniform convergence is also proved.

In (Ostrogorski, 1995; Ostrogorski, 1997a; Ostrogorski, 1997b; Ostrogorski, 1988), the research started in (Bajšanski, Karamata, 1969) is continued. As the group G, various cones in $\mathbf{R}^n$ are considered, such as the hyper-octant, the future light cone, arbitrary homogeneous cones.

In (Omey, 1989), measurable functions $f: \mathbf{R}_+^2 \to \mathbf{R}_+$ are studied such that the limit

$$\lim_{t\to\infty} \frac{f(r(t)x, s(t)y)}{f(r(t), s(t))} = \lambda(x, y)$$

exists for some auxiliary functions $r, s: \mathbf{R}_+ \to \mathbf{R}_+$, $r(t) \to \infty$, $s(t) \to \infty$ as $t \to \infty$, some positive function $\lambda(x, y)$ and for all $x, y > 0$.

In (Meerschaert, 1986; Meerschaert, 1988), functions $f(t)$ of one variable t are considered whose values are non-singular linear operators from $\mathbf{R}^k$, and the idea of regular variation is extended to this case.

In (Molchanov, 1993), regularly varying functions $f(x)$ defined in some m-dimensional cone are introduced whose values are closed (compact) sets in $\mathbf{R}^d$.

According to (Resnick, 1986), a random vector X taking values in $\mathbf{R}^n$ is said to be regularly varying at infinity with index $\alpha \geq 0$ and spectral (probability) distribution P_s on the unit sphere $S^{n-1} \subset \mathbf{R}^n$ if there exist positive c and σ_k, $k \in \mathbf{N}$, such that, as $k \to \infty$,

$$k\mathbf{P}\{\sigma_k^{-1} X \in A(r, B)\} \to cr^{-\alpha} P_s(B)$$

for all sets $B \subset S^{n-1}$ of continuity of the limiting measure P_s and $r > 0$, where

$$A(r, B) = \{x: x \in \mathbf{R}^n,\ |x| > r,\ x/|x| \in B\}.$$

In (Basrak *et al.*, 2002), it is shown that if a random vector X regularly varies at infinity with index $\alpha > 0$, then for any $x \in \mathbf{R}^n$ and some regularly varying at infinity function $L(t)$ there exists the limit

$$\lim_{t\to\infty} \frac{\mathbf{P}\{(x, X) > t\}}{t^{-\alpha} L(t)} = \omega(x),$$

and there exists $x_0 \neq 0$ such that $\omega(x_0) > 0$. It is also shown that for non-integer $\alpha > 0$ the corresponding converse assertion is true, while the limiting measure P_s is uniquely determined by the function $\omega(x)$. A counterexample is given for $\alpha = 2$.

Let a non-negative function $\lambda(x)$ be continuous on the unit sphere $S^{n-1} \subset \mathbf{R}^n$. In (Zaigraev, Nagaev, 2003), a function $f(x)$, $x \in \mathbf{R}^n$, is said to be (β, λ)-regularly varying if, as $|x| \to \infty$,

$$\sup_{e_x \in E_\lambda} \left| \frac{f(x)}{r_\beta(|x|)} - \lambda(e_x) \right| = o(1),$$

where $e_x = x/|x|$, $r_\beta(t)$ regularly varies as $t \to \infty$ with index β, and

$$E_\lambda = \{a \in S^{n-1}: \lambda(a) > 0\}.$$

This definition is close to the definition of the class $R_2(\Gamma)$ in Section 1.1, where

$$\Gamma = \{x \in \mathbf{R}^n,\ x = ta,\ a \in E_\lambda,\ t > 0\}.$$

So, if the set E_λ is closed, then the set of all measurable (β, λ)-regularly varying functions coincides with the set of functions in $R_2(\Gamma)$ such that the limiting homogeneous function $\varphi(x)$ in relation (1.1.1) is of the form $\varphi(x) = c|x|^\beta \lambda(x/|x|)$, $x \in \Gamma$, where c is a positive constant (see Theorems 1.1.1 and 1.1.2).

Prior to the seventies of the twentieth century, the multidimensional Tauberian theory was of little utility and followed the line of direct extension of known one-dimensional Tauberian theorems. Particular results concerning multidimensional Tauberian theorems of this period can be found in works of many mathematicians: (Hardy, Littlewood, 1913; Knopp, 1939; Ganelius, 1971; Delange, 1953; Delange, 1963; Delavault, 1961), etc. But most studies which deal with multidimensional Tauberian theorems came to the light in the second half of seventies of the twentieth century (see the references in Introduction) in connection with many applications in the theory of differential equations, mathematical physics, and probability theory.

Sections 1.1–1.3 are based on (Yakymiv, 1982). The studies (Yakymiv, 1988; Yakymiv, 1990a) underlie Sections 1.4 and 1.5. One-dimensional Tauberian theorems in Section 1.6 are taken from (Yakymiv, 1987a; Yakymiv, 1987b; Yakymiv, 1995; Yakymiv, 2002). In Section 1.7, we utilise (Drozhzhinov, Zavyalov, 1984; Drozhzhinov, Zavyalov, 1986a; Drozhzhinov, Zavyalov, 1986b; Drozhzhinov, Zavyalov, 1990; Drozhzhinov, Zavyalov, 1992; Drozhzhinov, Zavyalov, 1995a; Drozhzhinov, Zavyalov, 1995b) and the monograph (Vladimirov *et al.*, 1988). Section 1.8 follows (Yakymiv, 2003b).

2

Applications to branching processes

2.1. Bounded below branching processes

In this section we consider one-dimensional continuous-time *Markov branching processes* homogeneous in time. We recall that Markov branching processes are distinguished from the other Markov processes with phase space $\mathbf{N} = \{0, 1, 2, \dots\}$ by the property of branching transition probabilities $\{P_{ij}(t), i, j \in \mathbf{N}, t \in \mathbf{R}_+\}$:

$$P_{ij}(t) = P_{1j}^{*i}(t) = \sum_{j_1+j_2+\cdots+j_i=j} P_{1j_1}(t)P_{1j_2}(t)\cdots P_{1j_i}(t).$$

The last condition for $i = 0$ means that

$$P_{0j}(t) = P_{1j}^{*0}(t) = \delta_{0j},$$

where δ_{ij} are the Kronecker symbols,

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

As is the convention for branching processes, the state of the process is called the number of particles. Let a Markov branching process $\mu(t)$ be given. We set

$$P_n(t) = \mathsf{P}\{\mu(t) = n \mid \mu(0) = 1\} = \mathsf{P}\{\mu(t+u) = n | \mu(u) = 1\}.$$

As usual,

$$\lim_{t\to 0} P_1(t) = 1,$$

and, as $t \to 0$,

$$P_1(t) = 1 + p_1 t + o(t), \quad P_n(t) = p_n t + o(t), \quad n \neq 1,$$

while

$$\sum_{n=0}^{\infty} p_n = 0.$$

We set

$$f(s) = \sum_{n=0}^{\infty} p_n s^n, \qquad s \in [0, 1].$$

We assume that $f'(1) = 0$ (the process is *critical*) and $b = f''(1) < \infty$. As we know (Sevastyanov, 1974, Section 2.2), in this case the probability that the process continues is

$$Q(t) = \mathbf{P}\{\mu(t) > 0 \mid \mu(0) = 1\}$$

behaves at infinity as follows:

$$Q(t) \sim \frac{2}{bt}, \qquad t \to \infty.$$

In (Sevastyanov, 1978), the asymptotic behaviour at infinity of the probabilities

$$Q_{mr}(t) = \mathbf{P}\{\inf_{0 \le u \le t} \mu(u) > r \mid \mu(0) = m\}, \qquad m > r \ge 0, \tag{2.1.1}$$

is studied. It is clear that $Q_{10}(t) = Q(t)$, $Q_{m0}(t) = 1 - (1 - Q(t))^m \sim mQ(t)$ as $t \to \infty$. Therefore,

$$Q_{m0}(t) \sim \frac{2m}{bt}, \qquad t \to \infty.$$

In (Sevastyanov, 1974), the following theorem is proved.

THEOREM 2.1.1. *Let $f'(1) = 0$, $f''(1) = b \in (0, \infty)$. Then for any integers $m > r \ge 0$*

$$Q_{mr}(t) \sim \frac{2(m-r)}{bt}, \qquad t \to \infty. \tag{2.1.2}$$

We give here the original proof of this theorem.

In order to prove the theorem, we make use of the equality

$$\tau_{m0} = \tau_{mr} + \tau_{r0}, \tag{2.1.3}$$

where τ_{m0} (τ_{r0}) is the random time of extinction of the process beginning with m (r) particles, τ_{mr} is the random time of first passage of r for the process beginning with m particles, τ_{mr} and τ_{r0} are independent. Let $\varphi_{mr}(\lambda)$ denote the Laplace transform of τ_{mr}:

$$\varphi_{mr}(\lambda) = \mathbf{E}e^{-\lambda\tau_{mr}}, \quad \lambda \ge 0.$$

We see that

$$\varphi_{mr}(\lambda) = -\int_0^{\infty} e^{-\lambda t} dQ_{mr}(t) = 1 - \lambda\Phi_{mr}(\lambda),$$

where

$$\Phi_{mr}(\lambda) = \int_0^{\infty} e^{-\lambda t} Q_{mr}(t)\, dt.$$

From (2.1.3) it follows that

$$\varphi_{mr}(\lambda) = \frac{\varphi_{m0}(\lambda)}{\varphi_{r0}(\lambda)}, \tag{2.1.4}$$

or, in other notation,

$$\Phi_{mr}(\lambda) = \frac{\Phi_{m0}(\lambda) - \Phi_{r0}(\lambda)}{1 - \lambda\Phi_{r0}(\lambda)}. \tag{2.1.5}$$

In what follows we will make use of one Tauberian theorem which extends Theorem 4 in (Feller, 1966, Section XIII.5). We formulate it as a lemma.

LEMMA 2.1.1. *Let*

$$\omega(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\, dt, \qquad u(t) = t^\gamma v(t) \geq 0,$$

where $v(t)$ *is a monotone function; let* $\rho > 0$ *and* $L(t)$ *be a slowly varying function. Then*

$$\omega(\lambda) \sim \frac{1}{\lambda^\rho} L\left(\frac{1}{\lambda}\right), \qquad \lambda \downarrow 0,$$

if and only if

$$u(t) \sim \frac{1}{\Gamma(\rho)} t^{\rho-1} L(t), \qquad t \to \infty.$$

Lemma 2.1.1 is a corollary to Theorem 1.3.4; it can also be proved similarly to that Tauberian theorem in (Feller, 1966).

PROOF OF THEOREM 2.1.1. Since $Q_{m0}(t) = 1 - (1 - Q(t))^m$, we see that

$$\Phi_{m0}(\lambda) = m\Phi(\lambda) + O(1),$$

where

$$\Phi(\lambda) = \int_0^\infty e^{-\lambda t} Q(t)\, dt. \tag{2.1.6}$$

We cannot apply Lemma 2.1.1 to (2.1.6), so we integrate (2.1.6) by parts and obtain

$$\Phi(\lambda) = \lambda \int_0^\infty e^{-\lambda t} \widetilde{Q}(t)\, dt,$$

where

$$\widetilde{Q}(t) = \int_0^t Q(u)\, du.$$

Since

$$\widetilde{Q}(t) \sim \frac{2}{b} \ln t, \qquad t \to \infty,$$

by Lemma 2.1.1 we obtain

$$\Phi(\lambda) \sim \frac{2}{b} \ln \frac{1}{\lambda}, \qquad \lambda \downarrow 0. \tag{2.1.7}$$

From this asymptotic expression with account for (2.1.5) it follows that

$$\Phi_{mr}(\lambda) \sim \frac{2(m-r)}{b} \ln \frac{1}{\lambda}, \qquad \lambda \downarrow 0. \tag{2.1.8}$$

We cannot apply Lemma 2.1.1 to the asymptotic formula (2.1.8), so we differentiate (2.1.5) and obtain

$$\Phi'_{mr}(\lambda) = \frac{\Phi'_{m0}(\lambda) - \Phi'_{r0}(\lambda)}{1 - \lambda\Phi_{r0}(\lambda)} + \frac{(\Phi_{mo}(\lambda) - \Phi_{r0}(\lambda))(\Phi_{r0}(\lambda) + \lambda\Phi'_{r0}(\lambda))}{(1 - \lambda\Phi_{r0}(\lambda))^2}, \tag{2.1.9}$$

where

$$\Phi'_{mr}(\lambda) = -\int_0^\infty e^{-\lambda t} t Q_{mr}(t)\, dt.$$

We observe that

$$\Phi'_{m0}(\lambda) = -m\int_0^\infty e^{-\lambda t} t Q(t)\, dt + \binom{m}{2}\int_0^\infty e^{-\lambda t} t Q^2(t)\, dt + O(1).$$

By virtue of Lemma 2.1.1, as $\lambda \downarrow 0$

$$\int_0^\infty e^{-\lambda t} t Q(t)\, dt \sim \frac{2}{b\lambda},$$
$$\int_0^\infty e^{-\lambda t} t Q^2(t)\, dt \sim \frac{4}{b^2}\ln\frac{1}{\lambda}.$$

Therefore,

$$\Phi'_{m0}(\lambda) \sim -\frac{2m}{b\lambda}, \qquad \lambda \downarrow 0.$$

Thus, from (2.1.9) with account for the asymptotic expression (2.1.8) we obtain

$$\Phi'_{mr}(\lambda) \sim -\frac{2(m-r)}{b\lambda}, \qquad \lambda \downarrow 0. \tag{2.1.10}$$

In order to complete the proof it remains to apply Lemma 2.1.1 to (2.1.10). □

REMARK 2.1.1. In the case of discrete time, jumps down by several units may occur, so equation (2.1.3) is broken and the problem to find an asymptotic formula remains open.

The example below demonstrates that in the case of discrete time an asymptotic formula differing from (2.1.2) may arise. Let the offspring generating function be of the form

$$F(s) = P_0 + P_{r+1}s^{r+1} + \cdots, \qquad P_0 > 0, P_{r+1} > 0.$$

In this case,

$$Q_{mr}(t) = Q_{m0}(t) \sim mQ(t), \qquad t \to \infty.$$

The direction considered in this section is continued to be studied in (Sevastyanov, 1995; Sevastyanov, 1997); both the case of discrete time and subcritical branching processes with several types of particles have been analysed. Even more general problem is posed in (Sevastyanov, 1998), where branching processes are considered which stop after falling into a prescribed set of states. Most interesting results concern the case of subcritical discrete-time branching processes. The initial multi-type Galton–Watson branching process

$$\mu(t) = (\mu_1(t), \dots, \mu_m(t)), \qquad t = 0, 1, 2 \dots,$$

generates a stopped branching process $\xi(t)$ if the process stops as soon as $\mu(t)$ finds itself in a certain finite set S. The process $\mu(t)$ is assumed to be critical and indecomposable (Sevastyanov, 1978, Chapter IV). It is proved that

$$q_r^n = \lim_{t\to\infty} \mathbf{P}\{\xi(t) = r \mid \xi(0) = n\}$$

for any $r \in S$ as $n \notin S$, $\overline{n} = n_1 + \cdots + n_m \to \infty$, $n_i/\overline{n} \to \alpha_i$, asymptotically approximates a periodic with period one function of $\log_{1/R} \overline{n}$, R is the Perron root of the matrix $\{A_{ij} = \mathbf{E}\mu_i(1) \mid \mu(0) = e_i\}$ (Sevastyanov, 1978, Chapter IV, Section 5).

2.2. Bellman–Harris branching processes

In this section and two following ones we consider a more general model of branching processes, the so-called *Bellman–Harris processes*. In these processes, every particle is an independent probabilistic copy of the initial particle whose life time distribution function is $G(t)$, $t \geq 0$, and the generating function of the number of particles which a particle generates when dying is $h(s)$, $s \in [0, 1]$. As before, we consider *critical* branching processes, that is, $h'(1) = 1$ (on average, a particle generates a single particle). For more details, the reader should refer to (Harris, 1963, Chapter 4) and to (Sevastyanov, 1974, Chapters 8 and 9, model 3).

Here we study a new phenomenon in these processes which does not take place in Markov branching processes. This phenomenon was discovered in (Vatutin, 1977b). His study, with only a slight modification, is presented below.

Let $\mu(t)$ be the number of particles at time t in a Bellman–Harris branching process under the condition that at the initial time $t = 0$ the process has a single particle of zero age. In this section, $L(t)$, $L_0(t)$, $L_1(t)$ stand only for functions which slowly vary at infinity. In (Vatutin, 1977b), the following limit theorem is proved.

THEOREM 2.2.1. *Let* $G(0_+) = 0$,

$$h(s) = s + (1-s)^{1+\alpha} L\left(1/(1-s)\right), \qquad s \in (0,1), \tag{2.2.1}$$

$$T(t) \equiv 1 - G(t) = t^{-\beta} L_0(t), \qquad t > 0, \tag{2.2.2}$$

$$\frac{n(1-G(n))}{1-h_n(0)} \to \infty, \qquad n \to \infty, \tag{2.2.3}$$

where $\alpha \in (0, 1]$, $\beta > 0$, $h_n(s)$ *is the nth iteration of the generating function* $h(s)$. *Then*

(1) $\mathbf{P}\{\mu(t) > 0\} \sim t^{-\beta/(1+\alpha)} L_1(t)$ *as* $t \to \infty$;

(2) *for any* $s \in [0, 1)$,

$$\lim_{t\to\infty} \mathbf{E}\left[s^{\mu(t)} | \mu(t) > 0\right] = 1 - (1-s)^{1/(1+\alpha)}; \tag{2.2.4}$$

as is our convention, the functions L, L_0, L_1 *are slowly varying at infinity.*

Let us discuss the theorem. Constraint of form (2.2.1) on the generating function $h(s)$ was first used in (Zolotarev, 1957) for Markov branching processes (that is, where either

$G(t)$ is concentrated at 1 or $G(t) = 1 - \exp(-\rho t)$, $t \geq 0$, for some $\rho > 0$). It is proved there that for the Markov branching processes obeying (2.2.1) the conditional distribution $\mu(t)\mathbf{P}\{\mu(t) > 0\}$ under the condition that $\mu(t) > 0$ weakly converges as $t \to \infty$ to the stable distribution with parameter α. In (Slack, 1968; Slack, 1972), it is shown that (2.2.1) is necessary and sufficient for existence of this weak limit for the Markov discrete-time branching processes. In addition, the asymptotic behaviour of $\mathbf{P}\{T = t\}$ was found there as $t \to \infty$ which related to the probability of continuation of the process $\mathbf{P}\{\mu(t) > 0\} = \mathbf{P}\{T > t\}$ as $t \to \infty$. Similar limit theorems are true for the Bellman–Harris branching processes if the ratio in (2.2.3) tends to zero (Vatutin, 1977a). The case where the ratio in (2.2.3) tends to a positive constant is analysed in (Vatutin, 1980). We observe that no normalisation of $\mu(t)$ enters into in (2.2.4), although

$$\mathbf{E}\{\mu(t) \mid \mu(t) > 0\} = \frac{1}{\mathbf{P}\{\mu(t) > 0\}} \to \infty, \qquad t \to \infty.$$

This is why Theorem 2.2.1 generates so much excitement among specialists on branching processes.

As mentioned in (Bojanić, Seneta, 1971), from (2.2.1) it follows that

$$1 - h_n(0) \sim n^{-1/\alpha} L_2(n), \qquad n \to \infty. \tag{2.2.5}$$

From (2.2.5) and (2.2.2) it follows that (2.2.7) holds if and only if either $\beta < 1 + 1/\alpha$, or $\beta = 1 + 1/\alpha$,

$$\lim_{n\to\infty} L_0(n)/L_2(n) = \infty.$$

In order to prove Theorem 2.2.1, we need a series of lemmas.

LEMMA 2.2.1 (see (Nagaev, 1975)). *Let $\mu_1(t)$ and $\mu_2(t)$ be Bellman–Harris branching processes, $G_1(t)$ and $G_2(t)$ be the distribution functions of the life times of a particle in the processes, and let the generating function $h(s)$ of the number of offspring of a particle be the same for both processes, $h'(1) = 1$. If*

$$G_1(t) \geq G_2(t)$$

for all $t \geq 0$, then

$$\mathbf{E}s^{\mu_1(t)} \equiv F_1(t,s) \geq F_2(t,s) \equiv \mathbf{E}s^{\mu_2(t)}$$

for all $s \in [0,1]$ and $t \geq 0$.

LEMMA 2.2.2 (see (Goldstein, 1971)). *If $h'(1) = 1$ in a Bellman–Harris branching process, then*

$$h_n(s) + (1-s)G^{*n}(t) \geq \mathbf{E}s^{\mu(t)} \geq h_n(s) - (1-s)(1 - G^{*n}(t)) \tag{2.2.6}$$

for all $t \geq 0$, $s \in [0,1]$, and all positive integers n. Here $G^{*n}(t)$ is the n-fold convolution of the function $G(t)$.

LEMMA 2.2.3. *Let (2.2.3) hold with $\beta > 1$. If $n \to \infty$, $t \to \infty$ so that $n = o(t)$, then*

$$1 - G^{*n}(t) = o(t^{-\beta+1} L_0(t)).$$

The proof of this lemma repeats the proof of Theorem 27 in (Petrov, 1975, Chapter IX), so we omit it.

LEMMA 2.2.4. *Let $L_1(t)$ and $L_2(t)$ be slowly varying at infinity. If, as $t \to \infty$,*

$$L_1(t) = o(L_2(t)), \tag{2.2.7}$$

then there exists a non-negative integer-valued function $n(t)$ such that, as $t \to \infty$,

$$n(t) \to \infty, \quad n(t) = o(t), \quad \frac{L_1(n(t))}{n(t)} = o\left(\frac{L_2(t)}{t}\right). \tag{2.2.8}$$

PROOF. From (2.2.7) and the definition of a slowly varying function it follows that for any $c > 0$

$$\lim_{t\to\infty} \frac{L_1([ct])}{ct} \frac{t}{L_2(t)} = 0.$$

Let

$$a_k(t) = kL_1([t/k])/L_2(t), \quad t_1 = 1, \quad t_{k+1} = 1 + \min\{t \ge t_k : a_k(x) \le 1/k \ \forall x \ge t\}.$$

The function $n(t)$, which is equal to 1 for $t \le 1$ and to $[t/k]$ for $t_k \le t < t_{k+1}$, obeys (2.2.8). □

PROOF OF THEOREM 2.2.1. As we know (Harris, 1963, Theorem 7.1), the function

$$F(t,s) = \mathbf{E}s^{\mu(t)}$$

satisfies the integral equation

$$F(t,s) = s(1 - G(t)) + \int_0^t h(F(t-u,s))\,dG(u),$$

which can be rewritten as follows:

$$R(t,s) = (1-s)T(t) + \int_0^t R(t-u,s)\,dG(u) - \int_0^t g(R(t-u,s))\,dG(u), \tag{2.2.9}$$

where

$$R(t,s) = 1 - F(t,s), \quad T(t) = 1 - G(t), \quad g(s) = h(1-s) - (1-s).$$

Applying the Laplace transformation to equation (2.2.9), we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t} R(t,s)\,dt\,\lambda \int_0^\infty e^{-\lambda t} T(t)\,dt \\ = (1-s)\int_0^\infty e^{-\lambda t} T(t)\,dt - \int_0^\infty e^{-\lambda t} g(R(t,s))\,dt \int_0^\infty e^{-\lambda t}\,dG(t).\end{aligned} \tag{2.2.10}$$

Since $R(t,s) \to 0$ as $t \to \infty$ for any fixed s, as $\lambda \downarrow 0$ we obtain

$$\lambda \int_0^\infty e^{-\lambda t} R(t,s)\,dt = o(1). \tag{2.2.11}$$

Let $\beta \in (0,1)$. Then by virtue of Lemma 2.1.1, as $\lambda \downarrow 0$

$$(1-s)\int_0^\infty e^{-\lambda t} T(t)\,dt \sim \frac{(1-s)\Gamma(1-\beta)}{\lambda^{1-\beta}} L_0(1/\lambda). \tag{2.2.12}$$

From (2.2.10) and (2.2.11) it follows that, as $\lambda \downarrow 0$,

$$(1-s)\int_0^\infty e^{-\lambda t} T(t)\,dt \sim \int_0^\infty e^{-\lambda t} g(R(t,s))\,dt. \tag{2.2.13}$$

Since $h'(1) = 1$, for any fixed s the function $R(t,s)$ monotonically decreases, and therefore, the function $g(R(t,s))$ monotonically decreases as well. Making use of Lemma 2.1.1, from (2.2.12) and (2.2.13) we obtain

$$g(R(t,s)) = (1-s)t^{-\beta} L_0(t)(1+o(1)), \qquad t \to \infty. \tag{2.2.14}$$

Setting $s = 0$ in (2.2.14) and taking (2.2.1) into account, we find that

$$R(t,0)^{1+\alpha} L(1/R(t,0)) = t^{-\beta} L_0(t)(1+o(1)), \qquad t \to \infty. \tag{2.2.15}$$

Hence it follows that $R(t,0) = t^{-\beta/(1+\alpha)} L_1(t)$, which proves the first part of the theorem for $\beta \in (0,1)$. Furthermore, because the function $F(t,s)$ is convex downwards,

$$R(t,0) \geq R(t,s) \geq (1-s)R(t,0).$$

So, for any $s \in [0,1)$

$$\lim_{t\to\infty} L(1/R(t,s))/L(1/R(t,0)) = 1.$$

Recalling (2.2.1) and (2.2.14), we arrive at

$$\lim_{t\to\infty} \left(\frac{R(t,s)}{R(t,0)}\right)^{1+\alpha} = \lim_{t\to\infty} \frac{g(R(t,s))}{g(R(t,0))} = 1-s,$$

which is equivalent to (2.2.4).

The following proof is carried out by induction. Let

$$u(n) = \sum_{k=0}^{n} (1+\alpha)^k, \qquad n = 0,1,\ldots$$

Let (2.2.14) be true for all $\beta \in (0, u(n))$. Let us prove that (2.2.14) is true for $\beta \in [u(n), u(n+1))$. First, from Lemma 2.2.1 and the induction assumption it follows that, as $t \to \infty$,

$$R(t,s) = o(t^{-(u(n)-\varepsilon)(1+\alpha)}), \tag{2.2.16}$$

$$g(R(t,s)) = o(t^{-u(n)+\varepsilon}) \tag{2.2.17}$$

for any $\varepsilon > 0$ and any $\beta \geq u(n)$. To see this, let the distribution function $G_1(t)$ be chosen so that $1 - G_1(t) \sim t^{-u(n)+\varepsilon}$ as $t \to \infty$. Then, obviously, there exists M such that $1 - G(t) \leq 1 - G_1(t)$ for all $t \geq M$. It is clear that $G_1(t)$ can be re-defined on $[0, M]$ in such a way that $1 - G(t) \le 1 - G_1(t)$ for all $t \ge 0$. Making use of Lemma 2.2.1 and the induction assumption, we arrive at (2.2.16) and (2.2.17).

By differentiating equality (2.2.10) $\nu = [\beta]$ times with respect to λ, we arrive at

$$(-1)^\nu \int_0^\infty e^{-\lambda t} t^\nu R(t,s)\,dt\,\lambda \int_0^\infty e^{-\lambda t} T(t)\,dt + \sum_{k=0}^{\nu-1} \binom{\nu}{k} (-1)^k \int_0^\infty e^{-\lambda t} t^k R(t,s)\,dt$$
$$\times \left((-1)^{\nu-k} \lambda \int_0^\infty e^{-\lambda t} t^{\nu-k} T(t)\,dt + (\nu-k)(-1)^{\nu-k-1} \int_0^\infty e^{-\lambda t} t^{\nu-k-1} T(t)\,dt \right)$$
$$= (-1)^\nu (1-s) \int_0^\infty e^{-\lambda t} t^k T(t)\,dt - (-1)^\nu \int_0^\infty e^{-\lambda t} t^\nu g(R(t,s))\,dt \int_0^\infty e^{-\lambda t}\,dG(t)$$
$$- \sum_{k=0}^{\nu-1} (-1)^k \int_0^\infty e^{-\lambda t} t^k g(R(t,s)) (-1)^{\nu-k} \int_0^\infty e^{-\lambda t} t^{\nu-k}\,dG(t), \quad (2.2.18)$$

which can be rewritten in a more simple form as follows:

$$A(s,\lambda) = (-1)^\nu B_1(s,\lambda) - (-1)^\nu B_2(s,\lambda) + B_3(s,\lambda). \quad (2.2.19)$$

We estimate the left-hand and right-hand sides of (2.2.19) separately. Let β be not integer. Then for $0 \le k \le \nu - 1$

$$\int_0^\infty t^k T(t)\,dt < \infty, \qquad \int_0^\infty t^{k+1}\,dG(t) < \infty, \quad (2.2.20)$$

and as $\lambda \downarrow 0$

$$\int_0^\infty e^{-\lambda t} t^\nu T(t)\,dt \sim \frac{\Gamma(\nu - \beta + 1)}{\lambda^{\nu-\beta+1}} L_0(1/\lambda). \quad (2.2.21)$$

Furthermore, in view of (2.2.16) we obtain

$$\int_0^\infty e^{-\lambda t} t^k R(t,s)\,dt < \infty, \qquad k < \frac{u(n)-\varepsilon}{1+\alpha} - 1, \quad (2.2.22)$$
$$\int_0^\infty e^{-\lambda t} t^k R(t,s)\,dt = o(\lambda^{((u(n)-\varepsilon)/(1+\alpha))-\nu}), \qquad \frac{u(n)-\varepsilon}{1+\alpha} - 1 < k \le \nu - 1. \quad (2.2.23)$$

Besides, as $\lambda \downarrow 0$

$$\int_0^\infty e^{-\lambda t} t^\nu R(t,s)\,dt = o(\lambda^{((u(n)-\varepsilon)/(1+\alpha))-\nu-1}). \quad (2.2.24)$$

Combining (2.2.20)–(2.2.24) and recalling (2.2.11), we see that, as $\lambda \downarrow 0$,

$$A(s,\lambda) = o(\lambda^{((u(n)-\varepsilon)/(1+\alpha))-\nu}) + O(1). \quad (2.2.25)$$

The same reasoning yields, as $\lambda \downarrow 0$,

$$B_3(s,\lambda) = o(\lambda^{u(n)-\nu-\varepsilon}) + O(1). \quad (2.2.26)$$

Now we observe that for non-integer $\beta \in [u(n), u(n+1))$

$$\nu - (u(n) - \varepsilon)/(1+\alpha) < \nu - \beta + 1, \qquad \nu - u(n) + \varepsilon < \nu - \beta + 1$$

for ε small enough. Hence, with account for (2.2.18), (2.2.19), and (2.2.21), it follows that, as $\lambda \downarrow 0$,

$$(1-s)\int_0^\infty e^{-\lambda t} t^\nu T(t)\,dt \sim \int_0^\infty e^{-\lambda t} t^\nu g(R(t,s))\,dt.$$

Applying Lemma 2.1.1 to the last relation, we arrive at (2.2.14) for non-integer $\beta \in [u(n), u(n+1))$. For integer $\beta \in [u(n), u(n+1))$, the proof is similar with regard for the fact that, as $\lambda \downarrow 0$,

$$\int_0^\infty e^{-\lambda t} t^{\nu-1} T(t)\,dt \sim L_3(1/\lambda),$$
$$\int_0^\infty e^{-\lambda t} t^\nu \,dG(t) \sim L_4(1/\lambda).$$

From (2.2.14) in the same way as in the case $\beta \in (0,1)$ we find that the theorem is true. Since

$$\lim_{n\to\infty} u(n) = 1 + \alpha^{-1},$$

the above reasoning is applicable to all $\beta \in [1, 1+1/\alpha)$. If $\beta > 1 + 1/\alpha$, then, as we have seen, (2.2.3) is broken. Thus, it remains to consider the case $\beta = 1 + 1/\alpha$. If $\beta = 1 + 1/\alpha$, then from (2.2.2), (2.2.3), and (2.2.5) it follows that $L_2(t)/L_0(t) \to 0$ as $t \to \infty$. In accordance with Lemma 2.2.4 we choose $n = n(t)$ so that

$$n(t) \to \infty, \quad n(t) = o(t), \quad \frac{L_2(n(t))}{n(t)} = o\left(\frac{L_0(t)}{t}\right).$$

Recalling (2.2.5) and Lemma 2.2.3, we see that, as $t \to \infty$,

$$0 \le R(t,s) \le R(t,0) \le 1 - h_n(0) + 1 - G^{*n}(t) = o(t^{-1/\alpha} L_2(t)).$$

Using the last bound instead of (2.2.16) from (2.2.19) we derive (2.2.14) in the same way as for $\beta \in (1, 1+1/\alpha)$. The proof of the theorem is thus complete. □

2.3. Convergence of finite-dimensional distributions

As in Section 2.2, we assume that $\mu(t)$ is the number of particles in a critical Bellman–Harris branching process whose distribution function of the life time of a particle is $G(t)$ and whose generating function of the number of direct descendants of a particle is $h(s)$. Under the hypotheses of Theorem 2.2.1, we arrive here at a limit theorem for conditional finite-dimensional distributions of the process.

THEOREM 2.3.1. *Let the hypotheses of Theorem* 2.2.1 *be fulfilled. Then the finite-dimensional distributions of the process* $\{\mu(\tau t), \tau \in (0,1)\}$ *under the condition that*

$\mu(t) > 0$ converge, as $t \to \infty$, to the finite-dimensional distributions of some stochastic process $\{\eta(\tau),\ \tau \in (0,1)\}$, and

$$\mathbf{E}\prod_{i=1}^{n} s_i^{\eta(\tau_i)} = \left(\sum_{i=1}^{n}\left(\tau_i^{-\beta}(1-s_i)\prod_{j=1}^{i-1} s_j\right) + \prod_{i=1}^{n} s_i\right)^{\gamma} - \left(\sum_{i=1}^{n}\left(\tau_i^{-\beta}(1-s_i)\prod_{j=1}^{i-1} s_j\right)\right)^{\gamma}$$

for all $s_1,\dots,s_n \in [0,1]$, $0 < \tau_1 < \dots < \tau_n < 1$, where $\gamma = 1/(1+\alpha)$.

We observe that

$$\mathbf{E}s^{\eta(\tau)} = (\tau^{-\beta}(1-s)+s)^{\gamma} - (\tau^{-\beta}(1-s))^{\gamma} \to 0, \qquad \tau \to 0,$$

for $s \in (0,1)$ and $\tau \in (0,1)$, hence

$$\eta(\tau) \xrightarrow{\mathbf{P}} \infty, \qquad \tau \to 0.$$

This points to the fact that in the class of Bellman–Harris branching processes under consideration there should be ‘many’ particles at time τ as $\tau \to \infty$, $\tau = o(t)$, under the condition that the process does not extinct at time t. It is easily seen, indeed, that the corresponding limit theorem (Vatutin, Sagitov, 1991) holds true. In (Vatutin, Sagitov, 1988b), critical decomposable Bellman–Harris branching processes with several types of particles, which are ‘far away’ from Markov ones, are also investigated. Theorem 2.3.1 is proved in (Yakymiv, 1984).

In order to prove Theorem 2.3.1, we need the following four lemmas.

LEMMA 2.3.1. *For some $\beta > 0$ and a slowly varying at infinity function $L_0(t)$, let*

$$T(t) \equiv 1 - G(t) = t^{-\beta}L_0(t), \qquad t > 0,$$

that is, let (2.2.2) *hold, and let a function $f(t)$ be regularly varying at infinity with index $\rho > \beta$ and be integrable with respect to G in any finite interval inside* $[0,\infty)$. *Then, as $t \to \infty$,*

$$\int_0^t f(u)\,dG(u) \sim \frac{\beta}{\rho-\beta}T(t)f(t).$$

LEMMA 2.3.2. *Let a function $r(t)$ be defined for $t \ge 0$, do not increase, and be regularly varying at infinity so that $r(t) \to 0$ as $t \to \infty$. Then there exists a slowly varying at infinity function $l(t)$, $l(t) \to 0$ as $t \to \infty$, such that for all $t \ge 0$*

$$\int_0^t r(u)\,du \le tl(t).$$

LEMMA 2.3.3. *Let a function $T(t) = 1 - G(t)$ be weakly oscillating at infinity, that is, $T(\tau)/T(t) \to 1$ as $t \to \infty$, $\tau = t + o(t$, let a function $r(t) > 0$ be defined for $t \ge 0$, do not increase, and $r(t) \to 0$ as $t \to \infty$. Then for any $c \in (0,1)$*

$$I(t) = \int_{ct}^t r(t-u)\,dG(u) = o(T(t)), \qquad t \to \infty.$$

We set

$$q(t) = \mathbf{P}\{\mu(t) > 0\}, \qquad t \geq 0.$$

LEMMA 2.3.4. *Let the hypotheses of Theorem* 2.2.1 *be fulfilled. Then, as* $t \to \infty$,

$$q(t) = o(tT(t)).$$

PROOF OF LEMMA 2.3.1. Without loss of generality (Theorem 1.1.3), we assume that $f(t)$ is differentiable, while

$$f'(t) \sim \rho f(t)/t, \qquad t \to \infty. \tag{2.3.1}$$

Then, by integrating by parts, with the use of (2.3.1) and the well-known property of an integral of a regularly varying function (Seneta, 1976, Theorem 2.1), we obtain

$$\begin{aligned}
\int_0^t f(u)\,dG(u) = -\int_0^t f(u)\,dT(u) &= -f(u)T(u)|_0^t + \int_0^t T(u)f'(u)\,du \\
&= -T(t)f(t) + (1+o(1))\frac{t}{\rho-\beta}T(t)f'(t) + O(1) \\
&= -T(t)f(t) + (1+o(1))\frac{\rho}{\rho-\beta}T(t)f(t) \\
&= (1+o(1))\frac{\beta}{\rho-\beta}T(t)f(t), \qquad t \to \infty,
\end{aligned}$$

which is the desired result. □

PROOF OF LEMMA 2.3.2. The relations

$$\begin{aligned}
\int_0^t r(x)\,dx = \int_0^{\ln(1+t)} r(x)\,dx + \int_{\ln(1+t)}^t r(x)\,dx &\leq r(0)\ln(1+t) + tr(\ln(1+t)) \\
= t\left(\frac{r(0)\ln(1+t)}{t} + r(\ln(1+t))\right) &= tl(t)
\end{aligned}$$

are true. The function $l(t)$ just defined by the last relation satisfies all hypotheses of the lemma. The lemma is thus proved. □

PROOF OF LEMMA 2.3.3. We fix some $\varepsilon \in (0, 1-c)$. Then

$$I(t) = \int_{ct}^{t(1-\varepsilon)} r(t-u)\,dG(u) + \int_{t(1-\varepsilon)}^t r(t-u)\,dG(u) = I_1(t) + I_2(t).$$

For $I_1(t)$, the elementary estimate

$$I_1(t) \leq r(t\varepsilon)T(ct) = o(T(t)), \qquad t \to \infty,$$

is true. Further,

$$\begin{aligned} I_2(t) &= -\int_{t(1-\varepsilon)}^{t} r(t-u)\,dT(u) = -\,r(t-u)T(u)\big|_{t(1-\varepsilon)}^{t} + \int_{t(1-\varepsilon)}^{t} T(u)\,dr(t-u) \\ &= T(t(1-\varepsilon))r(t\varepsilon) - T(t)r(0) - \int_0^{t\varepsilon} T(t-v)\,dr(v) \\ &= T(t(1-\varepsilon))r(t\varepsilon) - T(t)r(t\varepsilon) - \int_0^{t\varepsilon} (T(t-v)-T(t))\,dr(v). \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} \frac{I_2(t)}{T(t)} = \limsup_{t\to\infty} \left| \int_0^{t\varepsilon} \left(\frac{T(t-v)}{T(t)} - 1 \right) dr(v) \right| \le \limsup_{t\to\infty} \left(\frac{T(t(1-\varepsilon))}{T(t)} - 1 \right) r(0).$$

So, we obtain

$$\limsup_{t\to\infty} \frac{I_2(t)}{T(t)} \le \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \left(\frac{T(t(1-\varepsilon))}{T(t)} - 1 \right) r(0) = 0.$$

The lemma is proved. □

PROOF OF LEMMA 2.3.4. For $t \ge 1$, let $a(t) = 1 - h_{[t]}(0)$. Then relation (3) in (Vatutin, 1977a) and (2.2.15) yield, as $t \to \infty$,

$$g(a(t)) \sim \frac{a(t)}{\alpha t}, \qquad g(q(t)) \sim T(t).$$

Taking (2.2.3) into account, we see that, as $t \to \infty$,

$$\frac{g(q(t))}{g(a(t))} \sim \frac{T(t)t\alpha}{a(t)} \to \infty.$$

Then $q(t) = o(a(t))$ as $t \to \infty$, so that

$$\liminf_{t\to\infty} \frac{tT(t)}{q(t)} \ge \lim_{t\to\infty} \frac{tT(t)}{a(t)} = \infty,$$

which proves the lemma. □

PROOF OF THEOREM 2.3.1. We fix an arbitrary $\varepsilon \in (0,1)$ and set

$$\begin{aligned} \Gamma_1 &= \{t\colon t \ge 0\}, \\ \Gamma_n &= \{x\colon x = (x_1,\dots,x_n),\ 0 \le \varepsilon x_n \le x_1 \le x_2 \le \dots \le x_n\}, \qquad n \ge 2, \\ \mathbf{R}_+^n &= \{x\colon x = (x_1,\dots,x_n),\ x_i > 0\ \forall i = 1,\dots,n\}, \\ \Delta_n &= \{s\colon s = (s_1,\dots,s_n),\ s_i \in [0,1]\ \forall i = 1,\dots,n\}, \end{aligned}$$

$\sigma_n = \operatorname{int}\Gamma_n$, $C_n = \operatorname{int}\Gamma_n^*$, where Γ_n^* is the cone dual to the cone Γ_n (see the beginning of Section 1.3),

$$F_n(x,s) = 1 - R_n(x,s) = \mathbf{E} \prod_{k=1}^{n} s_k^{\mu(x_k)}, \qquad q(t) = R_1(t,0),$$

where $x = (x_1, \ldots, x_n) \in \Gamma_n$, $s \in \Delta_n$, $t \geq 0$. Let $L_n(f)(\lambda)$ and $l_n(U)(\lambda)$, $\lambda \in C_n$, denote, respectively, the Laplace transform of a function $f(x) \geq 0$ defined in Γ_n and of a measure U in $\mathbf{R}^n$ over the cone Γ_n:

$$L_n(f)(\lambda) = \int_{\Gamma_n} e^{-(\lambda,x)} f(x)\,dx, \qquad l_n(u)(\lambda) = \int_{\Gamma_n} e^{-(\lambda,x)}\,U(dx).$$

We fix some $n > \beta+1$ and $s = (s_1, \ldots, s_n)$, $s \in \Delta_n \backslash \{e_n\}$. By calculating the mathematical expectation of $\prod_{k=1}^n s_k^{\mu(x_k)}$ under fixed time of death of the initial particle and the number of its offspring, and then averaging, we arrive at the integral equation for $F_n(x, s)$

$$\begin{aligned} F_n(x,s) = \int_0^{x_1} & h(F_n(x - ue_n, s)\,dG(u) \\ & + s_1 \int_{x_1}^{x_2} h(F_{n-1}((x_2 - u, \ldots, x_n - u), (s_2, \ldots s_n)))\,dG(u) + \cdots \\ & + s_1 \cdots s_{n-1} \int_{x_{n-1}}^{x_n} h(F_1(x_n - u, s_n))\,dG(u) \\ & + s_1 \cdots s_n T(x_n), \end{aligned}$$

As shown in (Esty, 1975), $F_n(x, s)$ is monotone in each of the variables x_k, $k = 1, \ldots, n$. Setting $f(t) = 1 - h(1 - t)$, from the last equation we obtain

$$R_n(x,s) = \psi_n(x,s) + \int_0^{x_1} f(R_n(x - ue_n, s))\,dG(u) + \delta_1(x,s), \tag{2.3.2}$$

where

$$\begin{aligned} \psi_n(x,s) &= (1 - s_1)T(x_1) + \cdots + s_1 \cdots s_{n-1}(1 - s_n)T(x_n), \\ \delta_1(x,s) &= s_1 \int_{x_1}^{x_2} f(R_{n-1}((x_2 - u, \ldots, x_n - u), (s_2, \ldots, s_n)))\,dG(u) + \ldots \\ &\quad + s_1 \cdots s_{n-1} \int_{x_{n-1}}^{x_n} f(R_1(x_n - u, s_n))\,dG(u). \end{aligned}$$

Since

$$R_n(x,s) = g(R_n(x,s)) + f(R_n(x,s)),$$

where, as before, $g(t) = h(1 - t) - 1 + t$, from (2.3.2) it follows that

$$g(R_n(x,s)) = \psi_n(x,s) + \delta_1(x,s) + \delta_2(x,s), \tag{2.3.3}$$

where

$$\delta_2(x,s) = \int_0^{x_1} f(R_n(x - ue_n, s))\,dG(u) - f(R_n(x,s)).$$

From (2.3.3) for $\lambda \in C_n$ we derive

$$L_n(g(R_n))(\lambda) = L_n(\psi_n)(\lambda) + L_n(\delta_1)(\lambda) + L_n(\delta_2)(\lambda) \tag{2.3.4}$$

(the Laplace transforms are over x). We observe that

$$\begin{aligned}
&L_n\left(\int_0^{x_1} f(R_n(x-ue_n,s))\,dG(u)\right)\\
&=\int_0^\infty dG(u)\int_u^\infty e^{-\lambda_1 x_1}dx_1\int_{x_1}^{cx_1} e^{-\lambda_n x_n}dx_n\int_{x_1}^{x_n} e^{-\lambda_2 x_2}dx_2\\
&\qquad\times\cdots\times\int_{x_{n-2}}^{x_n} e^{-\lambda_{n-1}x_{n-1}} f(R_n(x-ue_n,s)\,dx_{n-1}\\
&=\int_0^\infty e^{-\nu u}\,dG(u)\int_0^\infty e^{-\lambda_1 v_1}dv_1\int_{v_1}^{cv_1+bu} e^{-\lambda_n v_n}dv_n\int_{v_1}^{v_n} e^{-\lambda_2 v_2}dv_2\\
&\qquad\times\cdots\times\int_{v_{n-2}}^{v_n} e^{-\lambda_{n-1}v_{n-1}} f(R_n(v,s))\,dv_{n-1}\\
&=l_1(G)(\nu)L_n(f(R_n))(\lambda)+I(\lambda), \qquad (2.3.5)
\end{aligned}$$

where $c=1/\varepsilon$, $b=c-1$, $\nu=\lambda_1+\cdots+\lambda_n$, $v=(v_1,\ldots,v_n)$,

$$\begin{aligned}
I(\lambda)=\int_0^\infty e^{-\nu u}dG(u)\int_0^\infty e^{-\lambda_1 v_1}dv_1\int_{cv_1}^{cv_1+bu} e^{-\lambda_n v_n}dv_n\int_{v_1}^{v_n} e^{-\lambda_2 v_2}dv_2&\\
\times\cdots\times\int_{v_{n-2}}^{v_n} e^{-\lambda_{n-1}v_{n-1}} f(R_n(v,s))\,dv_{n-1}& \qquad (2.3.6)
\end{aligned}$$

(we change variables as follows: $v_k=x_k-u$, $k=1,\ldots,n$). Everywhere below we assume that $\lambda\in\mathbf{R}_+^n$. By virtue of (2.3.6),

$$I(\lambda)\le\int_0^\infty e^{-\nu u}\,dG(u)\int_0^\infty e^{-\lambda_1 v_1}\,dv_1\int_{cv_1}^{cv_1+bu}(v_n-v_1)^{n-2}r(v_1)\,dv_n,$$

where $r(t)=f(q(t))$ (because $R_n(v,s)\le R_n(v,0)=q(v_1)$). Since $(v_n-v_1)\le b(v_1+u)$ in the domain of integration, we see that

$$I(\lambda)\le C\int_0^\infty e^{-\nu u}u\,dG(u)\int_0^\infty e^{-\xi x}r(x)(u+x)^{n-2}\,dx$$

where $C=b^{n-1}$, $\xi=\lambda_1$. Dividing the integration domain in the last integral into two parts corresponding to the inequalities $x\le u$ and $x>u$, we obtain

$$I(\lambda)\le I_1(\lambda)+I_2(\lambda), \qquad (2.3.7)$$

where

$$\begin{aligned}
I_1(\lambda)&=C\int_0^\infty e^{-\nu u}u\,dG(u)\int_0^u e^{-\xi x}r(x)(u+x)^{n-2}\,dx,\\
I_2(\lambda)&=C\int_0^\infty e^{-\nu u}u\,dG(u)\int_u^\infty e^{-\xi x}r(x)(u+x)^{n-2}\,dx.
\end{aligned}$$

We estimate I_1 and I_2 separately:

$$\begin{aligned}I_1(\lambda) &\le C2^{n-2}\int_0^\infty e^{-\nu u}u^{n-1}\,dG(u)\int_0^u r(x)\,dx\\ &\le C2^{n-2}\int_0^\infty e^{-\nu u}u^n l(u)\,dG(u),\end{aligned}$$

where the function $l(u)$ is chosen in accordance with Lemma 2.3.2. By Lemma 2.3.1, in view of our choice of n, as $t\to\infty$ we find that

$$I_1(\lambda/t) = O(t^nT(t)l(t)) = o(t^nT(t)). \tag{2.3.8}$$

Let us turn to estimating the integral I_2:

$$\begin{aligned}I_2(\lambda) &= C\int_0^\infty e^{-\nu u}u\,dG(u)\int_u^\infty e^{-\xi x}r(x)(u+x)^{n-2}\,dx\\ &\le C2^{n-2}\int_0^\infty e^{-\nu u}u\,dG(u)\int_u^\infty e^{-\xi x}r(x)x^{n-2}\,dx\\ &\le C2^{n-2}\int_0^\infty e^{-\nu u}u\,dG(u)\int_0^\infty e^{-\xi x}r(x)x^{n-2}\,dx.\end{aligned}$$

Since $n > \beta + 1$ and by virtue of Lemma 2.3.4

$$q(t) = o(tT(t)), \qquad t\to\infty, \tag{2.3.9}$$

if $\int_0^\infty u\,dG(u) < \infty$ and $t\to\infty$, then

$$I_2(\lambda/t) = O(q(t)t^{n-1}) = o(t^nT(t)). \tag{2.3.10}$$

In the case where $\int_0^\infty u\,dG(u) = \infty$, for some slowly varying at infinity function $l_0(t)$ we see that

$$\int_0^t u\,dG(u) = tT(t)l_0(t)$$

so that

$$\begin{aligned}I_2(\lambda/t) &= O\left(\int_0^t u\,dG(u)\right)O(t^{n-1}q(t))\\ &= t^nT(t)O(q(t)l_0(t)) = o(t^nT(t)), \qquad t\to\infty,\end{aligned} \tag{2.3.11}$$

because $q(t)$ in our case is regularly varying with negative index (Theorem 2.2.1). From inequalities (2.3.8), (2.3.10), and (2.3.11) it follows that

$$I(\lambda/t) = o(t^nT(t)), \qquad t\to\infty. \tag{2.3.12}$$

From (2.3.5) and (2.3.12) as $t\to\infty$ we arrive at

$$\begin{aligned}L_n(\delta_2)(\lambda/t) &= L_n(f(R_n))(\lambda/t)(1-l_1(G))(\nu/t) + o(t^nT(t))\\ &= O(L_n(q(x_1))(\lambda/t))(\nu/t)L_1(T)(\nu/t) + o(t^nT(t))\\ &= O(t^{n-1}q(t))L_1(T)(\nu/t) + o(t^nT(t)) = o(t^nT(t))\end{aligned} \tag{2.3.13}$$

(for $\beta > 1$ the last equality in (2.2.13) immediately follows from (2.3.9); for $\beta \le 1$ it is obvious). For $x \in \Gamma_n$, we see that

$$\delta_1(x,s) \le \int_{x_1}^{x_2} q(x_2-u)\,dG(u) + \cdots + \int_{x_{n-1}}^{x_n} q(x_n-u)\,dG(u)$$
$$\le \int_{\varepsilon x_2}^{x_2} q(x_2-u)\,dG(u) + \cdots + \int_{\varepsilon x_n}^{x_n} q(x_n-u)\,dG(u).$$

Therefore, by virtue of Lemma 2.3.3, for $x \in \Gamma_n$, as $|x| \to \infty$, we obtain

$$\delta_1(x,s) = o(T(|x|)).$$

Hence we obtain, as $t \to \infty$, with the use of Theorems 1.5.5 and 1.5.6,

$$L_n(\delta_1)(\lambda/t) = o(t^n T(t)). \tag{2.3.14}$$

By virtue of Theorem 1.5.6, $L_n(\psi_n)(\lambda/t)$ is, as $t \to \infty$, of the same order of magnitude as $t^n T(t)$. So, from (2.3.4), (2.3.13), and (2.3.14) it follows that

$$L_n(g(R_n))(\lambda/t) = (1+o(1))L_n(\psi_n)(\lambda/t), \qquad t \to \infty.$$

Hence, with the use of Theorem 1.5.4, we conclude that for any $x \in \sigma_n$, as $t \to \infty$,

$$g(R_n(tx,s)) = (1+o(1))\psi_n(tx,s). \tag{2.3.15}$$

Since $\varepsilon > 0$ is arbitrary, we see that (2.3.15) is true indeed for all $x = (x_1, \dots, x_n)$, $0 < x_1 < x_2 < \cdots < x_n$. From (2.3.15) it follows that

$$g(R_n(tx,s)) = (1+o(1))\varphi_n(x,s)T(t), \qquad t \to \infty,$$

where

$$\varphi_n(x,s) = \sum_{i=1}^{n} \left(\tau_i^{-\beta}(1-s_i) \prod_{j=1}^{i-1} s_j \right).$$

Thus,

$$R_n(tx,s) = (1+o(1))(\varphi_n(x,s))^{\gamma} q(t), \qquad t \to \infty. \tag{2.3.16}$$

For any $s \in \Delta_n \setminus \{e_n\}$, $0 < x_1 < x_2 < \cdots < x_n < 1$, $t > 0$, we see that

$$\mathbf{E}\left\{ \prod_{k=1}^{n} s_k^{\mu(tx_k)} \mid \mu(t) > 0 \right\} = \frac{R_{n+1}(t(x_1,\dots,x_n,1),(s_1,\dots,s_n,0)) - R_n(tx,s)}{q(t)}.$$

In view of (2.3.16),the last expression tends to $(\varphi_n(x,s) + s_1 \cdots s_n)^{\gamma} - (\varphi_n(x,s))^{\gamma}$ as $t \to \infty$. The theorem is proved. □

2.4. The number of long-living particles

As in Sections 2.2 and 2.3, in this section we consider critical Bellman–Harris branching processes. We preserve the notation of Section 2.3 with one exception: let $\mu(\tau, t)$ for $0 \le \tau \le t$ be the number of particles in the process which are alive at time τ and which will be alive at time t,

$$F_2(x, s) = 1 - R_2(x, s) = \mathsf{E} s^{\mu(x_1, x_2)},$$

where $x = (x_1, \dots, x_n) \in \sigma_n$, $s \in (0, 1]$. For $\mu(\tau, t)$, here we prove the following limit theorem.

THEOREM 2.4.1. *Let the hypotheses of Theorem* 2.2.1 *be satisfied, and* $c \in (0, 1]$. *Then for all* $s \in [0, 1]$, *as* $t \to \infty$,

$$\mathsf{E}\{s^{\mu(ct,t)} \mid \mu(t) > 0\} \to 1 - (1 - s)^{\gamma}.$$

We stress that the limit distribution of $\mu(ct, t)$ under the condition that $\mu(t) > 0$ as $t \to \infty$ does not depend on $c \in (0, 1]$ and coincides with the limit distribution of $\mu(t)$ under the same condition. This points to the fact that for any fixed $\varepsilon > 0$ for large t with probability close to one all particles that live at time t are of age exceeding $t(1 - \varepsilon)$. Thus, the process continues its life primarily at the expense of the 'long-standing' particles.

The theorem below is proved in (Yakymiv, 1984). Further investigations in this direction are due to (Topchii, 1980; Topchii, 1982; Topchii, 1986; Topchii, 1987a; Topchii, 1987b; Topchii, 1988a; Topchii, 1988b; Topchii, 1988c; Topchii, 1990; Topchii, 1991; Sagitov, 1983; Sagitov, 1986a; Sagitov, 1986b; Sagitov, 1989; Sagitov, 1995). Topchii studied the asymptotic behaviour of critical Crump–Mode–Jagers branching processes $Z(t)$. In particular, he established a series of limit assertions concerning processes with long-living particles (when $\mathsf{P}\{Z(t) > 0\} \sim \mathsf{P}\{X(t) > 0\}$ as $t \to \infty$, where $X(t)$ is the number of particles in the process whose life times are at least t). Besides, he analysed the asymptotic behaviour of the probability of continuation of the processes $Z(t)$ as $t \to \infty$. Sagitov paid much attention to the asymptotic behaviour of reduced branching processes (we will present one of his results in the end of this section).

The main difference between the proofs of Theorems 2.4.1 and 2.3.1 consists of that the function $R_2(x, s)$ is not, generally speaking, monotone in x. Instead of monotonicity, we will show that $R_2(x, s)$ is q-slowly varying at infinity in σ_n. Namely, the following lemma is true.

LEMMA 2.4.1. *Let the hypotheses of Theorem* 2.2.1 *be satisfied. Then for all* $x_t, x \in \sigma_n$ *and* $s \in [0, 1]$ *as* $t \to \infty$ *and* $x_t \to x$

$$R_2(tx_t, s) - R_2(tx, s) = o(q(t)).$$

PROOF. Let us show that

$$\mathsf{P}\{\mu(\tau, t) \ne \mu(\tau, t_1)\} = o(q(t)) \tag{2.4.1}$$

as $\tau, t, t_1 \to \infty$, $t_1/t \to 1$, $\tau/t \to c \in (0, 1)$. Let $t_1 \ge t$, and let $d_1, \dots, d_{\mu(\tau)}$ be the particles which are alive at time τ, $\theta_1, \dots, \theta_{\mu(\tau)}$ be their ages at time τ, $\xi_1, \dots, \xi_{\mu(\tau)}$

be the times of their deaths, and let A be the least σ-algebra with respect to which $\mu(\tau), \theta_1, \dots, \theta_{\mu(\tau)}$ are measurable. Then

$$\begin{aligned}\mathbf{P}\{\mu(\tau,t) \neq \mu(\tau,t_1)\} &= \mathbf{EP}\{\exists i = 1,\dots,\mu(\tau) \colon \xi_i \in (t,t_1] \mid A\} \\ &= 1 - \mathbf{EP}\{\forall i = 1,\dots,\mu(\tau) \colon \xi_i \notin (t,t_1] \mid A\} \\ &= 1 - \mathbf{E}\prod_{i=1}^{\mu(\tau)} \mathbf{P}\{\xi_i \notin (t,t_1] \mid A\} \\ &= 1 - \mathbf{E}\prod_{i=1}^{\mu(\tau)} (1 - (T(t+\theta_i-\tau) - T(t_1+\theta_i-\tau))/T(\theta_i)) \\ &\leq R_1(\tau, 1-a),\end{aligned}$$

where

$$a = a(t,t_1,\tau) = \sup_{v\in[0,\tau]} \frac{T(t-\tau+v) - T(t_1-\tau+v)}{T(v)}.$$

In view of the above estimates, to prove (2.4.1) it remains to show that $a(t,t_1,\tau) \to 0$ as $t \to \infty, t_1/t \to 1, \tau/t \to c$. It is easily seen, indeed, that

$$\begin{aligned}\frac{T(t-\tau+v) - T(t_1-\tau+v)}{T(v)} &= \frac{T(t-\tau+v)}{T(v)}\left|1 - \frac{T(t_1-\tau+v)}{T(t-\tau+v)}\right| \\ &\leq \left|1 - \frac{T(t_1-\tau+v)}{T(t-\tau+v)}\right| \to 0\end{aligned}$$

as $t \to \infty$, $t_1/t \to 1$, $\tau/t \to c \in (0,1)$ uniformly in $v \in [0,\tau]$, because $T(t)$ regularly varies at infinity. Let is show that

$$\mathbf{P}\{\mu(\tau_1,t) \neq \mu(\tau_2,t)\} = o(q(t)) \tag{2.4.2}$$

as $t \to \infty$, τ_1/t, $\tau_2(t) \to c \in (0,1)$. Without loss of generality, let $\tau_1 \leq \tau_2$. Let $b_1,\dots,b_\eta$ be the particles which are alive at time τ_2 and born in the interval $(\tau_1,\tau_2]$, $v_1,\dots,v_\eta$ be their ages at time τ_2, $u_1,\dots,u_\eta$ be their life times, and B be the least σ-algebra with respect to which $\eta, v_1,\dots,v_\eta$ are measurable. Then

$$\begin{aligned}\mathbf{P}\{\mu(\tau_1,t) \neq \mu(\tau_2,t)\} &= \mathbf{EP}\{\exists i = 1,\dots,\eta \colon u_i \geq t-\tau_2+v_i \mid B\} \\ &= 1 - \mathbf{E}\prod_{i=1}^{\eta} \frac{G(t-\tau_2+v_i) - G(v_i)}{T(v_i)} \leq R_1(\tau_2, 1-b),\end{aligned}$$

where

$$b = b(\tau_1,\tau_2,t) = \sup_{0\leq v\leq \tau_2-\tau_1}\left(1 - \frac{G(t-\tau_2+v) - G(v)}{T(v)}\right),$$

because $\eta \leq \mu(\tau_2)$. But

$$1 - \frac{G(t-\tau_2+v) - G(v)}{T(v)} = \frac{T(t-\tau_2+v)}{T(v)} \leq \frac{T(t-\tau_2)}{T(\tau_2-\tau_1)} \to 0$$

as $t \to \infty$, $\tau_1/t, \tau_2(t) \to c \in (0,1)$, because

$$(t - \tau_2)/(\tau_2 - \tau_1) \to \infty.$$

Therefore, relation (2.4.2) holds. From (2.4.1) and (2.4.2) it follows that, as $t \to \infty$, $\tau/t \to c \in (0,1)$, $t_1/t \to 1$, $\tau_1/\tau \to 1$,

$$\mathbf{P}\{\mu(\tau_1, t_1) \neq \mu(\tau, t)\} = o(q(t)). \tag{2.4.3}$$

The lemma now follows from (2.4.3) and the definition of the function $R_2(x,s)$. □

PROOF OF THEOREM 2.4.1. Let τ be the time of death of the initial particle, and let ν be the number of particles it therewith produces. Calculating $\mathbf{E}\{s^{\mu(x_1,x_2)} \mid \tau, \nu\}$ and then averaging, for $x \in \Gamma_n$ and $s \in [0,1]$ we find that

$$\begin{aligned} F_2(x,s) = &\int_0^{x_1} h(F_2(x - ue_n, s))\, dG(u) \\ &+ \int_{x_1}^{x_2} (s(1 - h(1 - q(x_2 - u))) + h(1 - q(x_2 - u)))\, dG(u) + sT(x_2), \end{aligned}$$

hence we obtain

$$g(R_2(x,s)) = (1 - s)T(x_2) + \delta_1(x,s) + \delta_2(x,s),$$

where

$$\begin{aligned} \delta_1(x,s) &= (1 - s)\int_{x_1}^{x_2} f(q(x_2 - u))\, dG(u), \\ \delta_2(x,s) &= \int_0^{x_1} f(R_2(x - ue_n, s))\, dG(u) - f(R_2(x,s)). \end{aligned}$$

Repeating the proof of Theorem 2.3.1 word for word, for $\lambda \in \mathbf{R}_+^n$, $s \in [0,1)$, as $t \to \infty$ we obtain

$$L_n(g(r_2))(\lambda/t) = L_n(T(x_2))(\lambda/t)(1 - s)(1 + o(1)).$$

By the second assertion of Theorem 1.5.3, as $t \to \infty$

$$\frac{L_n(T(x_2))(\lambda/t)}{t^n T(t)} \to L_n(x_2^{-\beta})(\lambda).$$

Therefore,

$$\frac{L_n(g(R_2))(\lambda/t)}{t^n T(t)} \to L_n(x_2^{-\beta})(\lambda)(1 - s),$$

hence, by Theorem 1.5.9,

$$g(R_2(tx,s))/T(t) \to (1 - s)x_2^{-\beta}, \qquad t \to \infty,$$

or, for $x_2 = 1$, as $t \to \infty$,

$$g(R_2((tx_1, t), s) \sim (1 - s)T(t).$$

Since the function $g^{-1}(s)$, which is inverse to $g(s)$, is regularly varying with index $\gamma = 1/(1+\alpha)$, hence we conclude that, as $t \to \infty$,

$$R_2((tx_1, t), s) \sim g^{-1}((1-s)T(t)) \sim (1-s)^{\gamma} g^{-1}(T(t)) \sim (1-s)^{\gamma} q(t)$$

(the last follows from relation (2.2.15)). Therefore, for $c \in (0,1)$

$$\begin{aligned}\mathbf{E}\{s^{\mu(ct,t)} \mid \mu(t) > 0\} &= \frac{\mathbf{E}(s^{\mu(ct,t)}\chi\{\mu(t) > 0\})}{q(t)} = \frac{\mathbf{E}s^{\mu(ct,t)} - \mathbf{E}(s^{\mu(ct,t)}\chi\{\mu(t) = 0\})}{q(t)} \\ &= \frac{F_2((ct,t),s) - 1 + q(t)}{q(t)} = 1 - \frac{R_2((ct,t),s)}{q(t)} \to 1 - (1-s)^{\gamma}\end{aligned}$$

as $t \to \infty$. The theorem is thus proved. □

In conclusion, we present an intriguing result obtained under the hypotheses of Theorem 2.2.1 in (Sagitov, 1989). For $0 \le \tau \le t$, let $\nu(\tau, t)$ be the number of particles which are alive at time τ, die before time t, but have non-empty offspring at time t. We set

$$U(t) = \sum_{k=0}^{\infty} G^{*k}(t), \qquad t \ge 0.$$

Since relation (2.2.2) holds, $U(t)$ is regularly varying at infinity with index $\sigma = \min(1, \beta)$. We introduce the function

$$\theta(t) = \max\{\theta\colon (1+\alpha)U(\theta) = 1/\varphi(q(t))\},$$

where

$$\varphi(x) = \frac{h(1-x) - 1 + x}{x}, \qquad x \in (0, 1].$$

Roughly speaking, Sagitov's result consists of that the mentioned long-living particles appear in the time interval $[\varepsilon\theta, \varepsilon^{-1}\theta]$. More precisely, let us consider one more branching process with two types of particles T_1 and T_2. Let $\rho_i(t)$ be the number of particles at time t under the condition that one particle of type T_1 of zero age exists at time $t = 0$, and there is no other particle. As concerns the particles of type T_1, we assume that they form a Bellman–Harris branching process with the generating function of the number of direct descendants of a particle $s + \gamma(1-s)^{1+\alpha}$ and the distribution function of the life time of a particle $H(t)$ such that

$$\int_0^{\infty} e^{-\lambda t}\, dH(t) = \left(1 + \frac{\lambda^{\sigma}}{\Gamma(\sigma+1)}\right)^{-1}, \qquad \lambda \ge 0.$$

If a particle of type T_1, when dying, produces no particle of type T_1, then it transmutates into a particle of type T_2. The particles of type T_2 are not subject to change after birth. Thus, the process $(\rho_1(t), \rho_2(t))$ is a critical Bellman–Harris branching process with particles of final type (it is Markovian for $\sigma = 1$). For fixed $r_i, s_i \in [0,1)$, we set

$$f_n(t_1, \ldots, t_n) = \mathbf{E} r_1^{\rho_1(t_1)} \cdots r_n^{\rho_1(t_n)} s_1^{\rho_2(t_1)} \cdots s_n^{\rho_2(t_n)}.$$

In (Sagitov, 1989), the following theorem is proved.

THEOREM 2.4.2. *Let the hypotheses of Theorem* 2.2.1 *be satisfied. Then for* $\overline{\tau} = (\tau_1, \dots, \tau_n)$

$$\mathbf{E}\{r_1^{\nu(\tau_1,t)} \cdots r_n^{\nu(\tau_n,t)} s_1^{\mu(\tau_1,t)} \cdots s_n^{\mu(\tau_n,t)} \mid \mu(t) > 0\} = f_n\left(\frac{\overline{\tau}}{\theta}\right) + \varepsilon_n(t, \overline{\tau}),$$

where $\varepsilon_n(t, \overline{\tau}) \to 0$ *as* $t \to \infty$ *uniformly in* $0 \le \tau_1 \le \cdots \le \tau_n \le t$.

Along with the mentioned works, Tauberian theorems were used in studies related to branching processes in (Bingham, 1988; Bingham, Doney, 1974; Bojko, 1982; Bojko, 1983; Weiner, 1990; Seneta, 1969; Seneta, 1973; Seneta, 1974).

3

Random A-permutations

3.1. The number of A-permutations

It is well known that any permutation can be decomposed into cycles. For example, the permutation

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

has one cycle of length 1 (2 goes to 2) and one cycle of length 2 (1 passes to 3, while 3, to 1).

We fix some set $A \subseteq \mathbf{N} = \{1, 2, 3, \dots\}$. A permutation of degree n

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

is called an *A-permutation* (Sachkov, 1997, Section 5.0.2), if the lengths of the cycles in σ belong to the set A. Let $T_n = T_n(A)$ denote the set of all A-permutations of degree n. In our example, $s \in T_3(A)$ if $1, 2 \in A$. Let ζ_{nm} denote the *number of cycles* in a random permutation uniformly distributed on T_n which are of *length* $m \in A$, ζ_n be the *total* number of its cycles:

$$\zeta_n = \sum_{m \in A} \zeta_{nm}.$$

In this chapter, $|X|$ stands for the number of elements of a finite set or for the Lebesgue measure of an infinite set X.

In this chapter, we find the asymptotic behaviour of $|T_n|$ as $n \to \infty$, as well as the asymptotic behaviour (in the weak sense) of ζ_{nm} and ζ_n as $n \to \infty$ for fixed $m \in A$ for various kinds of sets A.

In the course of solution of these problems, we go from simple to complex. In this section, we will find the asymptotic behaviour of $|T_n|$ as $n \to \infty$ under the assumption that the set A is of unit asymptotic density:

$$|m\colon m \in A,\ m \le n|/n \to 1, \qquad n \to \infty. \tag{3.1.1}$$

In Section 3.2, we investigate the asymptotic behaviour (in the weak sense) of ζ_{nm} and ζ_n as $n \to \infty$ for fixed $m \in A$ under the same assumption (3.1.1). In Section 3.3, we consider the same problems but for sets A of positive asymptotic density. Sections 3.4 and 3.5 are of auxiliary nature: here we discuss examples of sets A for which the corresponding limit theorems are true. In Section 3.6, we consider the case of a random set A.

We set

$$A(n) = \{m: m \in A,\ m \le n\}, \qquad B = \mathbf{N} \setminus A,$$
$$B(n) = \{m: m \in B,\ m \le n\}, \qquad l(n) = \sum_{m \in B(n)} 1/m,$$

for any $t \ge 1$

$$l(t) = l([t]), \quad L(t) = \exp(-l(t)), \quad g(z) = \sum_{n \in B} z^n/n, \quad z \in [0, 1].$$

As above, we write $f(x) \sim g(x)$ as $x \to a$, if $f(x)/g(x) \to 1$ as $x \to a$. The following theorem is true.

THEOREM 3.1.1. *Let* (3.1.1) *hold. Then*

$$|T_n| \sim n!\, L(n), \qquad n \to \infty. \tag{3.1.2}$$

In (Bender, 1974; Pavlov A., 1987), the cases are considered where B is finite and where the series $\sum_{m \in B} 1/m$ converges, respectively.

COROLLARY 3.1.1. *The sequence* $b(n) = |T_n|/n!$ *is slowly varying at infinity, that is,*

$$b([\lambda n])/b(n) \to 1, \qquad n \to \infty, \tag{3.1.3}$$

for any $\lambda > 0$ *if and only if* (3.1.1) *holds.*

Thus (see (Seneta, 1976, Section 1.5)), if (3.1.1) holds, then $|T_n|$ tends to infinity faster than $n^{-\varepsilon} n!$ for any fixed $\varepsilon > 0$.

In order to prove Theorem 3.1.1, we need five lemmas.

LEMMA 3.1.1. *Let* (3.1.1) *hold. Then the function* $L(n)$ *is slowly varying at infinity.*

LEMMA 3.1.2. *Let* (3.1.1) *hold and* $b(n) = |T_n|/n!$, $b(0) = 1$. *Then, as* $z \uparrow 1$,

$$\sum_{n \ge 0} b(n) z^n \sim \frac{1}{1-z} L(1/(1-z)). \tag{3.1.4}$$

LEMMA 3.1.3. *As* $z \uparrow 1$, *for some* $\alpha > 0$ *and a slowly varying at infinity function* $L(t)$ *let*

$$\sum_{n \ge 0} b(n) z^n \sim \frac{1}{(1-z)^\alpha} L(1/(1-z)), \tag{3.1.5}$$

where $b(n)$ *are some non-negative numbers, and as* $n \to \infty$, $k = o(n)$, *let*

$$b(n+k) - b(n) = o\left(\sum_{i=0}^{n+k} b(i)/n\right). \tag{3.1.6}$$

Then, as $n \to \infty$,

$$b(n) \sim n^{\alpha-1} L(n)/\Gamma(\alpha).$$

LEMMA 3.1.4. *Let* (3.1.5) *hold, and as* $n \to \infty$, $k = o(n)$, *let*

$$b(n+k) - b(n) = O\left(\sum_{i=0}^{n+k} b(i)/n\right).$$

Then, as $n \to \infty$,

$$b(n) = O(n^{\alpha-1} L(n)).$$

LEMMA 3.1.5. *Let some non-negative numbers* $b(n)$, *a slowly varying at infinity function* $L(t)$, *and* $\alpha > 0$ *be given. Relation* (3.1.5) *holds true if and only if*

$$\sum_{i=0}^{n} b(i) \sim n^{\alpha} L(n)/\Gamma(\alpha+1)$$

as $n \to \infty$.

PROOF OF LEMMA 3.1.1. Since the function $L(n)$ is monotone, it suffices to show that

$$\frac{L(2n)}{L(n)} \to 1, \qquad n \to \infty,$$

which is equivalent to

$$l(2n) - l(n) \to 0, \qquad n \to \infty. \tag{3.1.7}$$

We observe that

$$0 \le l(2n) - l(n) = \sum_{m>n,\ m\in B}^{2n} 1/m \le \frac{1}{n} \sum_{m>n,\ m\in B}^{2n} 1 \le \frac{|B(2n)|}{n} \to 0$$

by virtue of (3.1.1), which implies (3.1.7). The lemma is thus proved. □

PROOF OF LEMMA 3.1.2. By relation (6.3) in (Bender, 1974),

$$\sum_{n=0}^{\infty} b(n) z^n = \exp\left(-\sum_{n\in B} z^n/n\right)/(1-z), \qquad z \in [0,1). \tag{3.1.8}$$

So, in order to prove the lemma it suffices to show that, as $z \uparrow 1$,

$$g(z) = \sum_{m\in B} z^m/m = l(1/(1-z)) + o(1). \tag{3.1.9}$$

We set $n = [1/(1-z)]$ and see that

$$g(z) - l(1/(1-z)) = \Delta_1 - \Delta_2, \tag{3.1.10}$$

where

$$\Delta_1 = \sum_{m\in B,\ m>n} z^m/m, \qquad \Delta_2 = \sum_{m\in B(n)} (1-z^m)/m.$$

By (3.1.1) and the definition of n,

$$0 \le \Delta_2 \le \sum_{m\in B(n)} (-m \ln z)/m = -|B(n)| \ln z = -n \ln z o(1) = o(1), \qquad z \uparrow 1. \tag{3.1.11}$$

We fix an arbitrary $M \in \mathbf{N}$. For $\varepsilon = -\ln z$, the relations

$$\begin{aligned} 0 \le \Delta_1 &\le \frac{|B(nM)|}{n} + \sum_{m=nM+1}^{\infty} z^m/m \le \frac{|B(nM)|}{n} + \int_{nM}^{\infty} \frac{e^{-\varepsilon x}}{x}\,dx \\ &= \frac{|B(nM)|}{n} + \int_{\ln(\varepsilon n M)}^{\infty} e^{-y}\,dy \end{aligned}$$

are true (the change $y = \ln(\varepsilon x)$ is carried out in the former integral). Hence

$$\limsup_{z\uparrow 1} \Delta_1 \le \int_{\ln M}^{\infty} e^{-y}\,dy.$$

Since M is arbitrary, we see that

$$\Delta_1 = o(1), \qquad z \uparrow 1. \tag{3.1.12}$$

Now (3.1.9) follows from (3.1.10)–(3.1.12). The lemma is proved. □

Lemmas 3.1.3 and 3.1.4 follow from Theorems 1.5.7 and 1.5.8 respectively (it suffices to set $a(m,n) = b(n)$, $\alpha = 1$, $r(t) = t^{1+\gamma}L(t)/\Gamma(\gamma)$ in these theorems). Lemma 3.1.5 is the well-known Karamata theorem, whose various extensions are given in Section 1.3.

PROOF OF THEOREM 3.1.1. We set

$$b(n) = |T_n|/n!, \quad n \in \mathbf{N}, \quad b(0) = 1, \quad b(-1) = 0, \quad \Delta(n) = b(n) - b(n-1).$$

From (3.1.8) it follows that

$$\sum_{n=0}^{\infty} \Delta(n) z^n = \exp(-g(z)). \tag{3.1.13}$$

We differentiate (3.1.13) with respect to z and obtain

$$\sum_{n=0}^{\infty} n\Delta(n) z^n = -\sum_{n\in B} z^n \exp(-g(z)).$$

Hence, for any $n \ge 0$

$$n\Delta(n) = -\sum_{i\in B(n)} \Delta(n-i),$$

or

$$b(n) - b(n-1) = -\frac{1}{n} \sum_{j\in M(n)} \Delta(j), \tag{3.1.14}$$

where

$$M(n) = \{j : 0 \le j \le n,\ n - j \in B\}.$$

From (3.1.14) it follows that for any $k \ge 0$

$$\begin{aligned}
b(n+k) - b(n-1) &= -\sum_{i=0}^{k} \frac{1}{n+i} \sum_{j \in M(n+i)} \Delta(j) \\
&= -\sum_{i=0}^{k} \frac{1}{n+i} \sum_{j \in M(n+i)} (b(j) - b(j-1)) \\
&= -\sum_{j=0}^{l} b(j) \left(\sum_{i=0, i \in B(n-j)}^{k} \frac{1}{n+i} - \sum_{i=0, i \in B(n-j+1)}^{k} \frac{1}{n+i} \right) \\
&= -\sum_{j=0}^{l} b(j) \left(\sum_{i=0, i \in B(n-j)}^{k} \frac{1}{n+i} - \sum_{i=-1, i \in B(n-j)}^{k-1} \frac{1}{n+i+1} \right) \\
&= -\sum_{j=0}^{l} b(j) \left(\sum_{i=0}^{k-1} \frac{1}{(n+i)(n+i+1)} \right. \\
&\qquad \left. + \frac{\chi_B(n+k-j)}{n+k} - \frac{\chi_B(n-1-j)}{n} \right),
\end{aligned}$$

where $l = n + k$, $\chi_B(x) = 1$ for $x \in B$ and $\chi_B(x) = 0$ for $x \notin B$. Hence it follows that for all integers l, n, $0 \le n \le l$, there exists $\theta \in [-1, 1]$ such that

$$b(l) - b(n) = \theta(l + 2 - n)\frac{S_l}{n^2} - \frac{1}{n}\sum_{j=0}^{l} b(j)(\chi_B(l-j) - \chi_B(n-j)), \tag{3.1.15}$$

where

$$S_l = \sum_{j=0}^{l} b(j).$$

Thus, $b(l) - b(n) = O(S_l/n)$ as $n \to \infty$, $l - n = o(n)$. Therefore, from Lemmas 3.1.2 and 3.1.4 it follows that, as $n \to \infty$,

$$b(n) = O(L(n)). \tag{3.1.16}$$

For any sequence of sets $U(n) \subseteq \{0, 1, 2, \ldots, n\}$ of asymptotic density zero (that is, $|U(n)|/n \to 0$ as $n \to \infty$) we easily see that

$$\sum_{j \in U(n)} L(j) = o(nL(n)), \tag{3.1.17}$$

because for any $\varepsilon > 0$

$$\sum_{j \in U(n)} L(j) = \Sigma_1^\varepsilon + \Sigma_2^\varepsilon,$$

where the summation in Σ_1^ε is over $j \in U(n)$, $j < n\varepsilon$, and in Σ_2^ε, over $j \in U(n)$, $j \geq n\varepsilon$, and, since

$$\Sigma_2^\varepsilon \leq |U(n)| \sup_{n\varepsilon \leq j \leq n} L(j) \sim |U(n)| L(n) = o(nL(n))$$

as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \sum_{j \in U(n)} L(j)/(nL(n)) \leq \lim_{n\to\infty} \sum_{j=0}^{[n\varepsilon]} L(j)/(nL(n)) = \varepsilon,$$

which implies (3.1.17) because ε is arbitrary. From (3.1.15), (3.1.16), and (3.1.17) it follows that $b(l) - b(n) = o(L(n))$ as $n \to \infty$ and $l - n = o(n)$. Karamata's theorem (Lemma 3.1.5) implies that $S_n \sim nL(n)$ as $n \to \infty$. In other words, as $n \to \infty$ and $l - n = o(n)$,

$$b(l) - b(n) = o(S_n/n). \tag{3.1.18}$$

With the use of Lemma 3.1.3, from (3.1.18) we find that $b(n) \sim L(n)$ as $n \to \infty$. The theorem is thus proved. □

PROOF OF COROLLARY 3.1.1. If (3.1.1) holds, then, by virtue of Theorem 3.1.1, $b(n) \sim L(n)$ as $n \to \infty$. By Lemma 3.1.1, $L(n)$ is slowly varying at infinity, so is the function $b(n)$. Vice versa, let the function $b(n)$ be slowly varying at infinity. As we have seen (formula (3.1.8)), the generating function for $b(n)$ is

$$\sum_{n=0}^{\infty} b(n) z^n = \exp\left(-\sum_{n \in B} z^n/n\right)/(1-z), \qquad z \in (0,1).$$

So, by Karamata's theorem (Lemma 3.1.5), the function $\exp\left(-\sum_{n\in B} z^n/n\right)$ must be slowly varying at 1. Thus, as $\lambda \downarrow 0$,

$$\sum_{n \in B} (e^{-\lambda n} - e^{-2\lambda n})/n \to 0.$$

Hence it follows that

$$\sum_{n \in B,\ n \leq 1/\lambda} e^{-\lambda n}(1 - e^{-\lambda n})/n \to 0, \qquad \lambda \downarrow 0. \tag{3.1.19}$$

But by virtue of the inequality $1 - \exp(-x) \geq x/e$, $x \in [0,1]$,

$$\sum_{n \in B,\ n \leq 1/\lambda} e^{-\lambda n}(1 - e^{-\lambda n})/n \geq \frac{\lambda}{e} \sum_{n \in B([1/\lambda])} e^{-\lambda n} \geq \lambda |B([1/\lambda])|/e^2.$$

So relation (3.1.19) yields

$$\lambda |B([1/\lambda])| \to 0, \qquad \lambda \downarrow 0,$$

which implies (3.1.1). The corollary is thus true. □

3.2. Auxiliary limit theorems

As in Section 3.1, here we assume that the set A has the unit asymptotic density:

$$|m: m \in A,\ m \le n|/n \to 1, \qquad n \to \infty. \tag{3.2.1}$$

Let ζ_{nm} stand for the number of cycles in a random permutation uniformly distributed on T_n of length $m \in A$, and let ζ_n be the total number of its cycles:

$$\zeta_n = \sum_{m \in A} \zeta_{nm}.$$

Let us prove the following two limit theorems.

THEOREM 3.2.1. *Let* (3.2.1) *hold. Then the distribution of the random variable*

$$\zeta_n' = \frac{\zeta_n - l(n)}{\sqrt{\ln n}}$$

weakly converges, as $n \to \infty$, to the standard normal law, where

$$l(n) = \ln n - \sum_{m \in B(n)} 1/m$$

(as above, $B = \mathbf{N} \setminus A$, $B(n) = \{m: m \in B,\ m \le n\}$).

THEOREM 3.2.2. *Let* (3.2.1) *hold. Then for any fixed $m \in A$ the distribution of the random variable ζ_{nm} weakly converges, as $n \to \infty$,. to the Poisson law with parameter* $1/m$.

The proof of these two theorems was first published in (Yakymiv, 1990a). Special cases of finite B and of converging series $\sum_{m \in B} 1/m$ were considered, respectively, in (Bender, 1974) and (Pavlov A., 1987). Theorems 3.2.1 and 3.2.2 extend also the known results obtained in (Goncharov, 1962). The case $A = \mathbf{N}$ is now investigated much better. The studies (Volynets, 1987; Volynets, 1988a; Volynets, 1988b; Pavlov A., 1988; Pavlov Yu., 1982; Pavlov Yu., 1988) and the monograph (Kolchin V., 1999) yield the complete spectrum of limit theorems for the probabilities $\mathsf{P}\{\zeta_n = m\}$ as $n \to \infty$ under various kinds of behaviour of $m = m(n)$. In (Timashov, 1998; Timashov, 2003), asymptotic expansion is given for these probabilities in powers of $1/m$ in the domain $1 < \alpha_0 \le n/m \le \alpha_1 < \infty$ as $n \to \infty$. Besides, in (Timashov, 2003) the probabilities $\mathsf{P}\{\zeta_n = m\}$, $\mathsf{P}\{\zeta_n \le m\}$, $\mathsf{P}\{\zeta_n \ge m\}$ are estimated in the domain $0 < \gamma_0 \le m/\ln n \le \gamma_1 < \infty$ with remainder term of order of magnitude $O((\ln n)^{-2})$.

COROLLARY 3.2.1. *As $n \to \infty$, let*

$$\sum_{m \in B(n)} 1/m = o(\sqrt{\ln n}).$$

Then Theorem 3.2.1 *remains true with $l(n)$ replaced by* $\ln n$.

Let us consider an example where the centring function in Theorem 3.2.1 cannot be replaced by $\ln n$. We set

$$B = \{j: j \in \mathbf{N},\ j = [i \ln^{\beta} i] \text{ for some } i \in \mathbf{N}\},$$

where $\beta \in (0, 1/2]$. Then, as $n \to \infty$,

$$\sum_{m \in B(n)} 1/m - (1-\beta)^{-1} \ln^{1-\beta}(n) \to c,$$

where c is some constant depending on β. So, Theorem 3.2.1 is true with $l(n)$ replaced by $\ln n - (1-\beta)^{-1} \ln^{1-\beta}(n)$ but not by $\ln n$. By virtue of Theorem 3.1.1, in this case

$$|T_n| \sim n!\, \exp(-(1-\beta)^{-1} \ln^{1-\beta}(n) - c), \qquad n \to \infty.$$

REMARK 3.2.1. If (3.2.1) holds, then

$$l(n) \sim \ln n, \qquad n \to \infty.$$

This follows from the fact that the logarithmic density of the set A exists and is equal to one in the case under consideration (Postnikov, 1988, Section 3.1).

Briefly speaking, we prove Theorems 3.2.1 and (3.2.2) in the following steps:

(1) With the use of Theorems 1.5.7 and 1.5.8, we find the asymptotic behaviour of the variables

$$a(m,n) = \frac{|T_n|}{n!} \mathbf{E} \exp(\zeta_n x / \sqrt{l(m)}) \tag{3.2.2}$$

as $m, n \to \infty$, x is fixed, where we set $1/\sqrt{l(m)} = 0$ for $l(m) \le 0$, which, by formula (0.13) in (Sachkov, 1997, Section 5.0.2), obey the relation

$$\sum_{n=0}^{\infty} a(m,n) v^n = \exp\left(\sum_{k \in A} \frac{v^k}{k} \exp(x/\sqrt{l(m)}) \right), \qquad v \in [0,1). \tag{3.2.3}$$

(2) From the asymptotic formula obtained for $a(m,n)$ we derive Theorem 3.2.1.

(3) From the equality

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{|T_n|}{n!} \mathbf{P}\{\zeta_{nm} = k\} x^k z^n = \exp\left(\sum_{k \in A} \frac{z^k}{k} + (x-1)\frac{z^m}{m} \right), \qquad x, z \in (0,1], \tag{3.2.4}$$

(see formula (0.14) in (Sachkov, 1997, Section5.0.2)) and Theorem 3.1.1 we arrive at Theorem 3.2.2.

We fix some real x and α, and set

$$f(t) = \sum_{m \in A} \frac{t^m}{m}, \quad g(t) = \sum_{m \in B} \frac{t^m}{m}, \quad F(t) = \exp(f(t)), \quad 0 \le t < 1,$$

$$h(n) = \exp(x/\sqrt{l(n)}), \quad L(n) = h(n) \exp\left(- \sum_{m \in B(n)} \frac{1}{m} + \frac{x^2}{2} \right),$$

$$r(n) = n^{1+\alpha} L(n) = n^{\alpha} \exp(l(n) + x^2/2) h(n), \quad n \in \mathbf{N}.$$

Here the functions $l(n)$, $L(n)$ differ from those introduced in Section 3.1.

LEMMA 3.2.1. *There exists $\alpha > 1$ such that for any $\lambda > 0$, as $n \to \infty$,*

$$\sum_{m=1}^{\infty} \exp(-\lambda m/n) m^{\alpha-1} \exp(l(n)h(m)) \sim \lambda^{-\alpha}\Gamma(\alpha) n^{\alpha} \exp(l(n) + x\sqrt{l(n)} + x^2/2). \tag{3.2.5}$$

PROOF. Without loss of generality, let $l(n) \geq 1$ for all $n \in \mathbf{N}$. We divide the sum in the left-hand side of relation (3.2.5) into the sums Σ_1, Σ_2, and Σ_3 over $m \in [1, n/\ln n)$, $m \in [n/\ln n, n \ln n]$, and $m \in (n \ln n, \infty)$ respectively. For $m \leq n$ we observe that there exists a constant $C < \infty$ depending on x such that

$$\begin{aligned} l(n)(h(m)-1) - x\sqrt{l(m)} &\leq \frac{x l(n)}{\sqrt{l(m)}} - x\sqrt{l(m)} + C\frac{l(n)}{l(m)} \\ &= \frac{x}{\sqrt{l(m)}}(l(n) - l(m)) + C\frac{l(n)}{l(m)} \\ &\leq |x| \ln(n/m) + C \ln(n/m) + C \end{aligned}$$

in view of the inequality

$$l(n) \leq l(m) + \ln(n/m), \qquad m \leq n. \tag{3.2.6}$$

Thus, setting $\beta = |x| + C$ and $\alpha = \beta + 1$, we obtain

$$l(n)(h(m)-1) \leq x\sqrt{l(m)} + C + \beta \ln(n/m).$$

Therefore, as $n \to \infty$,

$$\Sigma_1 = O\left(n^{\beta} e^{l(n)} \sum_{m=1}^{[n/\ln n]} \exp(x\sqrt{l(m)})\right) = o(n^{\alpha} \exp(l(n) + x\sqrt{l(n)}). \tag{3.2.7}$$

For $m \geq n$ we see that

$$l(n)h(m) \leq l(m) \exp(x\sqrt{l(m)}) = l(m) + x\sqrt{l(m)} + O(1) \tag{3.2.8}$$

as $n \to \infty$. We set $s = \exp(-\lambda/2)$. By (3.2.8), there exists a constant $C_1 < \infty$ such that

$$\begin{aligned} \Sigma_3 &\leq C_1 e^{-\lambda \ln(n)/2} \sum_{m \geq n \ln n} s^m m^{\beta} \exp(l(m) + x\sqrt{l(m)}) \\ &\leq C_1 n^{-\lambda/2} \sum_{m \geq 1} s^m m^{\beta} \exp(l(m) + x\sqrt{l(m)}) \\ &= o(n^{\alpha} \exp(l(n) + x\sqrt{l(n)})), \qquad n \to \infty. \end{aligned} \tag{3.2.9}$$

Let $m \in [n/\ln n, n \ln n]$. For some θ in this interval, in view of (3.2.6) and Remark 3.2.1, we see that

$$\begin{aligned} |\sqrt{l(n)} - \sqrt{l(m)}| &= \frac{1}{2}\frac{|l(n) - l(m)|}{\sqrt{l(\theta)}} \\ &\leq \frac{1}{2}\frac{|\ln(n/m)|}{\sqrt{l(\theta)}} = O\left(\frac{\ln \ln n}{\sqrt{l(n)}}\right) = o(1), \qquad n \to \infty. \end{aligned}$$

Therefore,

$$l(n)\exp(x/\sqrt{l(m)}) = l(n)\left(1+\frac{x}{\sqrt{l(m)}}+\frac{x^2}{2l(m)}+O(l^{-3/2}(m))\right)$$
$$= l(n)+x\sqrt{l(n)}+\delta(n,m),$$

where $\delta(n,m)\to 0$ as $n\to\infty$ uniformly in m. Hence it follows that

$$\Sigma_2 \sim \exp(l(n)+x\sqrt{l(n)}+x^2/2)\sum_{m\in[n/\ln n,n\ln n]} m^\beta\exp(\lambda m/n)$$
$$\sim \exp(l(n)+x\sqrt{l(n)}+x^2/2)\sum_{m=1}^{\infty} m^\beta\exp(\lambda m/n), \qquad n\to\infty.$$

In other words,

$$\Sigma_2 \sim (n/\lambda)^\alpha\Gamma(\alpha)\exp(l(n)+x\sqrt{l(n)}+x^2/2) \tag{3.2.10}$$

as $n\to\infty$. Relation (3.2.5) follows from (3.2.7), (3.2.9), and (3.2.10). The lemma is proved. □

PROOF OF THEOREMS 3.2.1 AND 3.2.2. 1. According to (3.2.3), for $v\in[0,1)$

$$\sum_{n=0}^{\infty} a(m,n)v^n = \exp(f(v)h(m)). \tag{3.2.11}$$

Therefore, for all $u,v\in[0,1)$

$$A(u,v)\equiv\sum_{m,n\geq 0} m^{\alpha-1}a(m,n)u^m v^n = \sum_{m=0}^{\infty} u^m m^{\alpha-1}\exp(h(m)f(v)), \tag{3.2.12}$$

where α is chosen in accordance with Lemma 3.2.1. For any fixed $\mu>0$, as $n\to\infty$, if we set $v=\exp(-\mu/n)$, we see that

$$f(v) = l(n)-\ln\mu+o(1). \tag{3.2.13}$$

It is easily seen indeed that, as $n\to\infty$,

$$f(v) = \ln(1-v)^{-1}-\sum_{m\in B} v^m/m = \ln(n/\mu)-\sum_{m\in B} v^m/m+o(1)$$
$$= l(n)-\ln\mu+\left(\sum_{m\in B(n)} 1/m-\sum_{m\in B} v^m/m\right)+o(1)$$
$$= l(n)-\ln\mu+\Delta_1-\Delta_2+o(1), \tag{3.2.14}$$

where

$$\Delta_1 = \sum_{m \in B(n)} (1 - \exp(-\mu m/n))/m,$$
$$\Delta_2 = \sum_{m \in B,\ m>n} \exp(-\mu m/n)/m.$$

For Δ_1, the following bound holds true as $n \to \infty$:

$$0 \le \Delta_1 \le \sum_{m \in B(n)} \left(\frac{\mu m}{n}\right)/m = \mu \frac{|B(n)|}{n} = o(1). \tag{3.2.15}$$

We fix an arbitrary $M \in \mathbf{N}$. The inequalities

$$0 \le \Delta_2 \le \frac{|B(nM)|}{n} + \sum_{m>nM} \exp(-\mu m/n)$$
$$\le \frac{|B(nM)|}{n} + \int_{nM}^{\infty} \exp(-\mu x/n)\, dx/x = \frac{|B(nM)|}{n} + \int_{\ln(\mu M)}^{\infty} \exp(-\exp(y))\, dy$$

are true (the change of variables $y = \ln(\mu x/n)$ is carried out in the former integral), which yields

$$\limsup_{n\to\infty} \Delta_2 \le \int_{\ln(\mu M)}^{\infty} \exp(-\exp(y))\, dy.$$

So, since M is arbitrary, we obtain

$$\Delta_2 = o(1), \qquad n \to \infty. \tag{3.2.16}$$

Now (3.2.13) follows from (3.2.14), (3.2.15), and (3.2.16). From (3.2.12), taking into account (3.2.13) and Lemma 3.2.1, we obtain, as $n \to \infty$,

$$A(e^{-\lambda/n}, e^{-\mu/n}) \sim \mu^{-1} \sum_{m=0}^{\infty} \exp(-\lambda m/n) m^{\alpha-1} \exp(l(n)h(m))$$
$$\sim \lambda^{-\alpha}\Gamma(\alpha)\mu^{-1} n^{\alpha} \exp(l(n) + x\sqrt{l(n)}) = \lambda^{-\alpha}\mu^{-1}\Gamma(\alpha) r(n) \tag{3.2.17}$$

for any fixed $\lambda, \mu > 0$. In other words, the function $A(u, v)$ obeys hypothesis (1.5.36) of Theorem 1.5.7 with $\gamma = 1$. We observe that the function $l(m)$ does not decrease, because for $m > n$

$$l(m) - l(n) = \ln(m/n) - \sum_{i \in B(m),\ i>n} 1/i \ge 0$$

in view of the inequality

$$\sum_{i=n+1}^{m} 1/i \le \int_n^m 1/x\, dx = \ln(m/n).$$

We fix $m \in \mathbf{N}$ and set

$$t = h(m), \quad a_n = a(m,n), \quad \Delta_n = a_n - a_{n-1}, \quad n \ge 0,$$

where $a_{-1} = 0$. By (3.2.11), for $z \in [0, 1)$

$$\sum_{n=0}^{\infty} a_n z^n = F(z)^t = (1-z)^{-t} \exp(-tg(z)). \tag{3.2.18}$$

Therefore, for any $z \in [0, 1)$

$$\sum_{n=0}^{\infty} \Delta_n z^n = (1-z)^{1-t} \exp(-tg(z)). \tag{3.2.19}$$

We differentiate (3.2.19) with respect to z and obtain

$$\begin{aligned}\sum_{n=0}^{\infty} n\Delta_n z^{n-1} &= -(1-t)(1-z)^{-t} \exp(-tg(z)) - tg'(z)(1-z)^{1-t} \exp(-tg(z)) \\ &= -(1-t)\sum_{n=0}^{\infty} a_n z^n - tg'(z)\sum_{n=0}^{\infty} \Delta_n z^n.\end{aligned}$$

Hence for any integer $n \geq 0$ we obtain

$$n\Delta_n = (t-1)a_{n-1} - t\sum_{i \in B(n)} \Delta_{n-i},$$

or

$$a_n - a_{n-1} = \frac{t-1}{n} a_{n-1} - \frac{t}{n} \sum_{j \in M(n)} \Delta_j, \tag{3.2.20}$$

where $M(n) = \{j : 0 \leq j < n,\ n-j \in B\}$. From (3.2.20) it follows that for any integer $k \geq 0$

$$a_{n+k} - a_{n-1} = (t-1)\sum_{i=0}^{k} \frac{a_{n+i-1}}{n+i} - t\sum_{i=0}^{k} \frac{1}{n+i} \sum_{j \in M(n+i)} \Delta_j. \tag{3.2.21}$$

As in the proof of (3.1.15), we find that for $l = n+k$

$$\begin{aligned}&\sum_{i=0}^{k} \frac{1}{n+i} \sum_{j \in M(n+i)} \Delta_j \\ &\qquad = \sum_{j=0}^{l} a_j \left(\sum_{i=0}^{k-1} \frac{1}{(n+i)(n+i+1)} + \frac{\chi_B(l-j)}{l} - \frac{\chi_B(n-1-j)}{n} \right),\end{aligned} \tag{3.2.22}$$

where

$$\chi_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B. \end{cases}$$

From (3.2.21) and (3.2.22) it follows that for any integers l, n, $0 < n < l$, there exist $\tau, \theta \in [-1, 1]$, such that

$$a_l - a_n = \tau|t-1|\frac{S_l}{n} + \theta(l-n)\frac{S_l}{n^2} - \frac{t}{n}\sum_{j=0}^{l} a_j(\chi_B(l-j) - \chi_B(n-j)), \qquad (3.2.23)$$

where

$$S_l = \sum_{j=0}^{l} a_j.$$

Thus, there exists a positive constant c_1 such that for all l, n, m, $l > n > 0$,

$$|a(m,l) - a(m,n)| \le \frac{c_1}{n}\sum_{i=0}^{l} a(m,i).$$

Therefore, by virtue of Theorem 1.5.8, $a(m,i) = O(L(m))$ as $m \to \infty$, $i \asymp m$. Hence it follows that for an arbitrary fixed $\varepsilon \in (0,1)$ and an arbitrary sequence of sets $U(n) \subseteq \{0,1,2,\dots,n\}$ of asymptotic density zero (that is, $|U(n)|/n \to 0$ as $n \to \infty$) as $n \to \infty$, $m \asymp n$

$$\sum_{i \in U_n,\ m\varepsilon \le i \le n} a(m,i) = o(nL(n)). \qquad (3.2.24)$$

Let $x \le 0$. Then $a(m,n)$ does not decrease in m. By (1.5.38),

$$\begin{aligned}\sum_{j=0}^{[m\varepsilon]} a(m,j) &\le \frac{1}{m^\alpha}\sum_{i=m+1}^{2m}\sum_{j=0}^{[m\varepsilon]} i^{\alpha-1}a(i,j)\\ &\sim \frac{1}{m^\alpha} r(m)(2^\alpha - 1)\varepsilon^\gamma/\alpha\gamma, \qquad m \to \infty. \end{aligned} \qquad (3.2.25)$$

In other words, the inequality

$$\limsup_{n\to\infty,\ m\asymp n} \sum_{i \in U(n)} a(m,i)/(nL(n)) \le \varepsilon c_2 \qquad (3.2.26)$$

holds (in our case $\gamma = 1$), where the constant c_2 depends on α only. Since ε is arbitrary, from (3.2.24) and (3.2.26) it follows that, as $n \to \infty$ and $m \asymp n$,

$$\sum_{i \in U(n)} a(m,i) = o(nL(n)). \qquad (3.2.27)$$

By (1.5.38), since $a(m,n)$ is monotone in m, we obtain

$$nL(n) \asymp \sum_{j=0}^{n} a(m,j), \quad n \to \infty, \quad m \asymp n. \qquad (3.2.28)$$

Let us prove this in detail. By virtue of (1.5.38), for any fixed $a > 0$ and $b > a$, as $n \to \infty$ and $m \asymp n$,

$$\sum_{i=[am]}^{[bm]} \sum_{j=0}^{n} i^{\alpha-1} a(i, j) \asymp r(n).$$

Therefore, since

$$i^{\alpha-1} \asymp m^{\alpha-1} \asymp n^{\alpha-1},$$

we obtain

$$\sum_{i=[am]}^{[bm]} \sum_{j=0}^{n} a(i, j) \asymp nr(n)/n^{\alpha} = n^2 L(n)$$

as $n \to \infty$ and $m \asymp n$. Since $a(m, j)$ is monotone, hence it follows that

$$\sum_{j=0}^{n} a(m, j) \le \frac{1}{m+1} \sum_{i=m}^{2m} \sum_{j=0}^{n} a(i, j) \asymp \frac{n^2 L(n)}{m} \asymp nL(n) \tag{3.2.29}$$

as $n \to \infty$ and $m \asymp n$. On the other hand, by the same token

$$\sum_{j=0}^{n} a(m, j) \ge \frac{1}{m - [m/2] + 1} \sum_{i=[m/2]}^{m} \sum_{j=0}^{n} a(i, j) \asymp nL(n) \tag{3.2.30}$$

as $n \to \infty$ and $m \asymp n$. From (3.2.29) and (3.2.30) we arrive at (3.2.28). Relations (3.2.23), (3.2.27), and (3.2.28) yield (1.5.37). Let us prove (1.5.37) in the case $x > 0$. Then $a(m,n)$ does not increase in m, and by (1.5.38), as $m \to \infty$, for fixed $\varepsilon > 0$

$$\begin{aligned} \sum_{j=0}^{[m\varepsilon]} a(m, j) &\le \left(\frac{2}{m}\right)^{\alpha} \sum_{i=[m/2]+1}^{m} \sum_{j=0}^{[m\varepsilon]} i^{\alpha-1} a(i, j) \\ &\sim 2^{\alpha}(1 - 2^{-\alpha}) \frac{\varepsilon}{\alpha} mL(m). \end{aligned}$$

Therefore, (3.2.27) holds in the case under consideration as well. Relation (3.2.28) also holds, because inequalities in (3.2.29) and (3.2.30) are merely inverted for $x > 0$. Thus, (1.5.37) is true for $x > 0$, too. So, the hypotheses of Theorem 1.5.6 are fulfilled. By virtue of this theorem, as $n \to \infty$,

$$a(ny, nz) \sim r(n) n^{-1-\alpha} = L(n)$$

for any fixed $y, z > 0$.

2. Upon setting $y = z = 1$ in the last relation, we obtain, as $n \to \infty$,

$$a(n, n) \sim L(n) = \exp\left(x\sqrt{l(n)} - \sum_{m \in B(n)} 1/m + x^2/2 \right). \tag{3.2.31}$$

From (3.2.2), taking (3.2.31) into account, we see that, as $n \to \infty$,

$$\frac{|T_n|}{n!}\mathbf{E}\exp(x\zeta_n/\sqrt{l(n)}) \sim \exp\left(x\sqrt{l(n)} - \sum_{m\in B(n)}\frac{1}{m} + \frac{x^2}{2}\right), \tag{3.2.32}$$

which with $x = 0$ again leads to Theorem 3.1.1. By virtue of Theorem 3.1.1, (3.2.32) takes the form

$$\mathbf{E}\exp(x\zeta_n/\sqrt{l(n)}) \sim \exp(x\sqrt{l(n)} + x^2/2), \qquad n \to \infty.$$

Thus, for any fixed real x, as $n \to \infty$,

$$\mathbf{E}\exp(x\zeta_n'') \to \exp(x^2/2),$$

where

$$\zeta_n'' = (\zeta_n - l(n))/\sqrt{l(n)}.$$

Making use of Curtiss' theorem, we conclude that Theorem 3.2.1 is true (see Remark 3.2.1).

3. From (3.2.4) it follows that for all $k \in \mathbf{N} \cup \{0\}$, $m \in A$, and $z \in [0, 1)$

$$\sum_{n=0}^{\infty}\frac{|T_n|}{n!}\mathbf{P}\{\zeta_{nm} = k\}z^n = \left(\frac{z^m}{m}\right)^k\frac{1}{k!}\exp\left(\sum_{i\in A,\ i\neq m}\frac{z^i}{i}\right). \tag{3.2.33}$$

From (3.2.33) it follows that

$$p(n)\mathbf{P}\{\zeta_{nm} = k\} = \sum_{\nu=0}^{n} q(\nu)p(n-\nu), \tag{3.2.34}$$

where

$$p(\nu) = \frac{|T_\nu|}{\nu!}, \qquad q(\nu) = \operatorname{coef}_{z^\nu}\left(\frac{z^m}{m}\right)^k\frac{1}{k!}\exp\left(-\frac{z^m}{m}\right). \tag{3.2.35}$$

A simple estimate for $q(\nu)$ follows from (3.2.25):

$$\begin{aligned} q(\nu) &= \frac{1}{m^k k!}\operatorname{coef}_{z^\nu}\sum_{\mu=0}^{\infty}\frac{z^{mk}(-z^m/m)^\mu}{\mu!} \\ &= O(m^{-\nu/m}/[\nu/m - k]!), \qquad \nu \to \infty. \end{aligned} \tag{3.2.36}$$

By virtue of Corollary 3.1.1, the sequence $p(n)$ is slowly varying at infinity. Therefore,

$$\sum_{\nu=0}^{[n/2]} q(\nu)p(n-\nu) \sim p(n)\sum_{\nu=0}^{[n/2]} q(\nu) \sim p(n)e^{-1/m}m^{-k}/k! \tag{3.2.37}$$

Further, in view of (3.2.36),

$$\sum_{\nu=[n/2]+1}^{n} q(\nu)p(n-\nu) \le \sum_{\mu=0}^{[n/2]} p(\mu) \max_{\nu \ge n/2} |q(\nu)|$$
$$= O(np(n)) \max_{\nu \ge n/2} |q(\nu)| = o(p(n)), \qquad n \to \infty. \qquad (3.2.38)$$

From (3.2.34), (3.2.37), and (3.2.38) it follows that

$$\mathbf{P}\{\zeta_{nm} = k\} = e^{-1/m} m^{-k}/k! + o(1)$$

as $n \to \infty$. Theorem 3.2.2 is proved. □

3.3. Fundamental limit theorems

We use the notation of the preceding section (except as otherwise noted). We intend to prove the following three theorems.

THEOREM 3.3.1. *As $n \to \infty$,*

(1) *let the relation*

$$|k : k \le n, k \in A|/n \to \sigma > 0, \qquad (3.3.1)$$

hold, that is, let the asymptotic density of the set A exist and be equal to σ;

(2) *for a constant $1 < C < \infty$, let*

$$|k : k \le n,\ k \in A,\ m-k \in A|/n \to \sigma^2 \qquad (3.3.2)$$

uniformly in $m \in [n, Cn]$.

Then

$$|T_n| \sim n!\, n^{\sigma-1} L(n) e^{-\sigma\gamma}/\Gamma(\sigma), \qquad n \to \infty,$$

where the function $L(n)$ is slowly varying at infinity, and

$$L(n) = \exp\left(\sum_{m \in A(n)} 1/m - \sigma \ln n\right),$$

where $A(n) = \{m : m \in A,\ m \le n\}$, γ is the Euler constant, $\Gamma(\cdot)$ is the gamma function.

THEOREM 3.3.2. *Let relations* (3.3.1) *and* (3.3.2) *hold. Then the distribution of the random variable $\zeta_n' = (\zeta_n - l(n))/\sqrt{\sigma \ln n}$ weakly converges, as $n \to \infty$, to the standard normal law, where*

$$l(n) = \sum_{m \in A(n)} 1/m.$$

THEOREM 3.3.3. *Let relations* (3.3.1) *and* (3.3.2) *hold. Then for any fixed $m \in A$ the distribution of the random variable ζ_{nm} weakly converges, as $n \to \infty$, to the Poisson law with parameter $1/m$.*

Theorems 3.3.1–3.3.3 are proved in (Yakymiv, 1991a).

Similar problems for another classes of sets A are considered in (Bolotnikov, Sachkov, Tarakanov, 1980; Bender, 1974; Volynets, 1985; Volynets, 1986; Volynets, 1989; Grusho, 1993; Ivchenko, Medvedev, 2002a; Kolchin A., 1994; Kolchin V., 1991; Kolchin V., 1992; Manstavicius, 2001; Mineev, Pavlov A., 1978; Mineev, Pavlov A., 1979; Pavlov A., 1985; Pavlov A., 1987; Pavlov A., 1988; Pavlov A., 1992; Pavlov A., 1995; Pavlov A., 1997).

More general combinatorial objects, random mappings with constraints on cycle lengths, are studied in (Sachkov, 1972; Sachkov, 1973) and the monograph (Sachkov, 1997). Let $S_n^{(1)}$ and $S_n^{(2)}$ be sets of permutations of degree n with odd and even lengths of cycles respectively. The cyclic structure of random permutations uniformly distributed on some factor-sets of these sets is considered in (Bolotnikov, Sachkov, Tarakanov, 1977; Bolotnikov, Tarakanov, 1977; Tarakanov, Chistyakov, 1976). Additive and multiplicative functions on permutations are investigated in (Manstavicius, 1996; Manstavicius, 2003). It is worthwhile to note that the set of solutions of the equation $x^d = e$ in the symmetric group S_n (e is the identity permutation) is a set of A-permutations, $A = \{d_0, \dots, d_r\}$, where $1 = d_0 < d_1 < \cdots < d_r = d$ are all distinct divisors of the number d. This is probably why so many mathematicians have taken an interest in A-permutations. The monograph (Kolchin V., 1999) contains a special chapter where the asymptotic properties of solutions of equations in an unknown permutation are studied. Now, the scheme known as the Ewens one (Ewens, 1972) is gaining acceptance, where all permutations are considered but the probability of occurrence of a particular permutation depends on the number of its cycles, see, e.g., (Arratia *et al.*, 1992; Donnelly *et al.*, 1991; Babu, Manstavicius, 2002; Babu, Manstavicius, 1999a; Babu, Manstavicius, 1999b; Ivchenko, Medvedev, 2001; Ivchenko, Medvedev, 2002a; Ivchenko, Medvedev, 2002b; Hansen, 1990). In the context of this scheme, the probability of occurrence of a permutation of degree n with k cycles is equal to $\theta^k/\theta(n)$, where $\theta(n) = \theta(\theta+1)\cdots(\theta+n-1)$ and θ is some positive real number. Let $S_{n,k}$ denote the set of permutations of degree n with k cycles. In many studies, the uniform distribution is defined on the set $S_{n,k}$, and then the cyclic structure of random permutations in $S_{n,k}$ is analysed, see (Kolchin V., 2002; Kazimirov, 2002; Kazimirov, 2003; Timashov, 2001; Cherepanova, 2003). Here we pointed out only a few problems of virtually unlimited field of research related to random permutations. Surveys on random permutations can be found in (Kolchin V., Chistyakov, 1976; Kolchin V., 1986; Kolchin V., 1999; Stepanov, 1969b; Stepanov, 1969a; Vershik, 1995).

REMARK 3.3.1. The second assertion of Theorem 3.3.1 is equivalent to the fact that (3.3.2) holds for any sequence $m = m(n)$ such that $m \geq n$ and $m = O(n)$ as $n \to \infty$.

In the next section, we give examples of sets A which obey (3.3.1) and (3.3.2). We fix some real x and $\alpha > 1$. For $t \geq 0$, we set

$$L_1(t) = \exp(x\sqrt{l(t)} + x^2/2 - \sigma\gamma)/\Gamma(\sigma),$$
$$r(t) = \exp(l(t))t^\alpha L_1(t) = t^{\alpha+\sigma}L(t)L_1(t) = t^{\alpha+\sigma}L_0(t),$$

where $L_0(t) = L(t)L_1(t)$ (for non-integer t, the functions are defined as at $[t]$).

In what follows, we need two lemmas given below.

LEMMA 3.3.1. *Let* (3.3.1) *be true. Then for any fixed* $\mu > 0$ *as* $n \to \infty$

$$f(e^{-\mu/n}) = l(n) - \sigma \ln \mu - \sigma\gamma + o(1). \tag{3.3.3}$$

LEMMA 3.3.2. *Let* (3.3.1) *and* (3.3.2) *be true. Then the numbers* $a(m,n)$ *defined by* (3.2.2) *obey* (1.5.37).

We prove Theorems 3.3.1 and 3.3.2 using Lemmas 3.3.1 and 3.3.2 as if they were valid.

PROOF OF THEOREMS 3.3.1 AND 3.3.2. Lemma 3.2.1 remains valid in our case as well, and its proof remains unchanged. In accordance with Lemma 3.2.1, we choose $\alpha > 1$ such that (3.2.5) holds. By (3.2.12), for all $u, v \in [0, 1)$

$$A(u,v) = \sum_{m\geq 0} u^m m^{\alpha-1} \exp(h(m) f(v)). \tag{3.3.4}$$

We set

$$u = \exp(-\lambda/t), \quad v = \exp(-\mu/t)$$

in (3.3.4), where $\lambda, \mu, t > 0$. By (3.3.3) and (3.3.4),

$$A(u,v) \sim \mu^{-\sigma} e^{-\sigma\gamma} \sum_{m\geq 0} u^m m^{\alpha-1} \exp(h(m) l(t))$$

as $t \to \infty$. Making use of Lemma 3.2.1, hence we obtain, as $t \to \infty$,

$$\begin{aligned} A(u,v) &\sim \lambda^{-\alpha} \mu^{-\sigma} \Gamma(\alpha) e^{-\sigma\gamma} t^\alpha \exp(l(t) + x\sqrt{l(t)} + x^2/2) \\ &= \lambda^{-\alpha} \mu^{-\sigma} \Gamma(\alpha)\Gamma(\sigma) t^\alpha \exp(l(t)) L_1(t) = \lambda^{-\alpha}\mu^{-\sigma}\Gamma(\sigma) r(t). \end{aligned}$$

From Lemma 3.3.2 and the last relation it follows that the hypotheses of Theorem 1.5.7 are fulfilled. As $n \to \infty$, this theorem yields

$$\begin{aligned} a(n,n) &\sim r(n) n^{-1-\alpha} = n^{\sigma-1} L(n) L_1(n) \\ &= n^{\sigma-1} L(n) \exp(x\sqrt{l(n)} + x^2/2 - \sigma\gamma)/\,\Gamma(\sigma). \end{aligned} \tag{3.3.5}$$

For $x = 0$, from (3.3.5) and (3.2.2) we obtain

$$p(n) = \frac{|T_n|}{n!} \sim n^{\sigma-1} L(n) e^{-\sigma\gamma}/\,\Gamma(\sigma), \qquad n \to \infty. \tag{3.3.6}$$

Theorem 3.3.1 is thus proved.

Next, from (3.2.2), (3.3.5), and (3.3.6) it follows that, as $n \to \infty$,

$$\mathbf{E} \exp(\zeta_n x/\sqrt{l(n)}) \sim \exp(x\sqrt{l(n)} + x^2/2),$$

or

$$\mathbf{E} \exp(x(\zeta_n - l(n))/\sqrt{l(n)}) \to \exp(x^2/2), \qquad n \to \infty.$$

To prove Theorem 3.3.2, it remains to say that under condition (3.3.1)

$$l(n) = \sum_{m\in A(n)} 1/m \sim \sigma \ln n, \qquad n \to \infty,$$

that is, in the case under consideration the logarithmic density of the set A

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{m\in A, m\leq n} 1/m$$

exists and is equal to σ (see (Postnikov, 1988, Section 3.1)). □

PROOF OF THEOREM 3.3.3. By virtue of relations (3.2.34), (3.2.35), and (3.2.36), for fixed $k \in \mathbf{N} \cup \{0\}$ and $m \in A$

$$p(n)\mathsf{P}\{\zeta_{nm} = k\} = \sum_{\nu=0}^{n} q(\nu)p(n-\nu), \tag{3.3.7}$$

where

$$p(\nu) = \frac{|T_\nu|}{\nu!},$$
$$q(\nu) = \mathrm{coef}_{z^\nu} \left(\frac{z^m}{m}\right)^k \frac{1}{k!} \exp\left(-\frac{z^m}{m}\right) = O(m^{-\nu/m}/[\nu/m-k]!), \qquad \nu \to \infty. \tag{3.3.8}$$

By virtue of Theorem 3.3.1, the sequence $p(n)$ is regularly varying at infinity. Therefore,

$$\sum_{\nu=0}^{[m\sqrt{n}]} q(\nu)p(n-\nu) \sim p(n) \sum_{\nu=0}^{[m\sqrt{n}]} q(\nu) \sim p(n)e^{-1/m}m^{-k}/k! \tag{3.3.9}$$

Furthermore, in view of (3.3.8) and Theorem 3.3.1, we obtain

$$\begin{aligned}\sum_{\nu=[m\sqrt{n}]+1}^{n} q(\nu)p(n-\nu) &\le \sum_{\mu=0}^{n} p(\mu) \max_{\nu \ge m\sqrt{n}} |q(\nu)| \\ &= O(np(n))O\left(\frac{m^{-\sqrt{n}}}{[\sqrt{n}-m]!}\right) = o(p(n)), \qquad n \to \infty.\end{aligned} \tag{3.3.10}$$

From (3.3.7), (3.3.9), and (3.3.10) it follows that, as $n \to \infty$,

$$\mathsf{P}\{\zeta_{nm} = k\} = e^{-1/m}m^{-k}/k! + o(1).$$

Theorem 3.3.3 is thus proved. □

PROOF OF LEMMA 3.3.1. First, let us prove that, as $\lambda \downarrow 0$,

$$f\left(e^{-\lambda}\right) = \sum_{n \in A(1/\lambda)} 1/n - \sigma\gamma + o(1). \tag{3.3.11}$$

We set

$$A'(n) = \bigcup_{m \in A(n)} [m, m+1].$$

For non-integer z, we set $A(z) = A([z])$ and $A'(z) = A'([z])$. As $\lambda \downarrow 0$, we see that

$$\sum_{n \in A(1/\lambda)} \frac{1-\exp(-\lambda n)}{n} = \int_{A'(1/\lambda)} \frac{1-\exp(-\lambda x)}{x}\, dx + o(1), \tag{3.3.12}$$

because

$$\left(\frac{1-\exp(-\lambda x)}{x}\right)' = \lambda\frac{\exp(-\lambda x)}{x} - \frac{1-\exp(-\lambda x)}{x^2}.$$

Then

$$\lambda \sum_{n\le 1/\lambda} \frac{\exp(-\lambda n)}{n} \le \lambda \sum_{n\le 1/\lambda} 1/n = O(\lambda \ln(1/\lambda)) = o(1), \qquad \lambda \downarrow 0,$$

and

$$\sum_{n\le 1/\lambda} \frac{1-\exp(-\lambda n)}{n^2} \le \lambda \sum_{n\le 1/\lambda} 1/n = o(1), \qquad \lambda \downarrow 0.$$

Next,

$$\int_{A'(1/\lambda)} \frac{1-\exp(-\lambda x)}{x}\,dx = \int_{\lambda A'(1/\lambda)} \frac{1-\exp(-y)}{y}\,dy = \int_0^1 f_\lambda(y)\,dy, \tag{3.3.13}$$

where

$$f_\lambda(y) = \begin{cases} (1-\exp(-y))/y, & y \in \lambda A'(1/\lambda), \\ 0, & y \notin \lambda A'(1/\lambda). \end{cases}$$

We set

$$\varphi(y) = (1-\exp(-y))/y.$$

The function $\varphi(y)$ monotonically decreases for $y \ge 0$:

$$\varphi'(y) = \frac{\exp(-y)}{y} - \frac{1-\exp(-y)}{y^2} = \frac{(y+1)\exp(-y)-1}{y^2} \le 0,$$

because $1 + y \le e^y$, $y \ge 0$, so for any a, b, $0 \le a < b \le 1$,

$$\varphi(b)|\lambda A'(1/\lambda) \cap [a,b]| \le \int_a^b f_\lambda(y)\,dy \le \varphi(a)|\lambda A'(1/\lambda) \cap [a,b]|.$$

Thus,

$$\limsup_{\lambda\downarrow 0} \int_a^b f_\lambda(y)\,dy \le \varphi(a) \lim_{\lambda\downarrow 0} |\lambda A'(1/\lambda) \cap [a,b]|,$$

which yields

$$\limsup_{\lambda\downarrow 0} \int_a^b f_\lambda(y)\,dy \le \varphi(a)\sigma(b-a). \tag{3.3.14}$$

Similarly we obtain

$$\varphi(b)\sigma(b-a) \le \liminf_{\lambda\downarrow 0} \int_a^b f_\lambda(y)\,dy.$$

We partition $[0, 1]$ into m equal parts by $0 = a_0, a_1, a_2, \dots, a_m = 1$. With the use of relation (3.3.14), we obtain

$$\limsup_{\lambda\downarrow 0} \int_0^1 f_\lambda(y)\,dy \le \sigma \sum_{i=0}^{m-1} \varphi(a_i)(a_{i+1} - a_i);$$

now, if we let m grow without bound and take into account the fact that the right-hand side contains the Darboux sum for the integral of the function $\varphi(x)$, then we find that

$$\limsup_{\lambda\downarrow 0} \int_0^1 f_\lambda(y)\,dy \le \sigma \int_0^1 \varphi(y)\,dy.$$

Similarly we obtain

$$\liminf_{\lambda\downarrow 0} \int_0^1 f_\lambda(y)\,dy \ge \sigma \int_0^1 \varphi(y)\,dy.$$

Thus, there exists

$$\lim_{\lambda\downarrow 0} \int_0^1 f_\lambda(y)\,dy = \sigma \int_0^1 \varphi(y)\,dy.$$

From (3.3.12), (3.3.13), and the last inequality it follows that

$$\sum_{n\in A(1/\lambda)} \frac{1-\exp(-\lambda n)}{n} \to \sigma \int_0^1 \frac{1-\exp(-y)}{y}\,dy, \qquad \lambda \downarrow 0. \tag{3.3.15}$$

We set

$$A''(z) = \bigcup_{m>[z],m\in A} [m, m+1], \qquad z \ge 0.$$

It is easily seen that

$$\sum_{n>1/\lambda, n\in A} \frac{\exp(-\lambda n)}{n} = \int_{A''(1/\lambda)} \frac{\exp(-y\lambda)}{y}\,dy + o(1) \tag{3.3.16}$$

as $\lambda \downarrow 0$, because

$$\left(\frac{\exp(-y\lambda)}{y}\right)' = -\lambda\frac{\exp(-y\lambda)}{y} - \frac{\exp(-y\lambda)}{y^2}.$$

In addition,

$$\lambda \sum_{n>1/\lambda} \frac{\exp(-n\lambda)}{n} \le \lambda \sum_{n\ge 0} \frac{\exp(-n\lambda)}{n} = \lambda \ln(1-\exp(-\lambda))^{-1} = o(1), \qquad \lambda \downarrow 0,$$

and

$$\sum_{n>1/\lambda} \frac{\exp(-n\lambda)}{n^2} \le \sum_{n>1/\lambda} \frac{1}{n^2} = o(1), \qquad \lambda \downarrow 0.$$

Further,

$$\int_{A''(1/\lambda)} \frac{\exp(-y\lambda)}{y}\,dy = \int_1^{\infty} g_\lambda(y)\,dy, \tag{3.3.17}$$

where

$$g_\lambda(y) = \begin{cases} \exp(-y)/y, & y \in \lambda A''(1/\lambda), \\ 0, & y \notin \lambda A''(1/\lambda). \end{cases}$$

As above, for any a, b, $1 \le a < b < \infty$,

$$\limsup_{\lambda \downarrow 0} \int_a^b g_\lambda(y)\,dy \le \sigma(b-a)\psi(a), \qquad \psi(y) = \exp(-y)/y.$$

Thus, for any fixed $M > 1$

$$\limsup_{\lambda \downarrow 0} \int_1^M g_\lambda(y)\,dy \le \sigma \int_1^M \psi(y)\,dy < \sigma \int_1^{\infty} \psi(y)\,dy.$$

Since

$$\int_M^{\infty} g_\lambda(y)\,dy \le \int_M^{\infty} \psi(y)\,dy,$$

we obtain

$$\limsup_{\lambda \downarrow 0} \int_1^{\infty} g_\lambda(y)\,dy \le \sigma \int_1^{\infty} \psi(y)\,dy + \int_M^{\infty} \psi(y)\,dy.$$

Since M is arbitrary, we conclude that

$$\limsup_{\lambda \downarrow 0} \int_1^{\infty} g_\lambda(y)\,dy \le \sigma \int_1^{\infty} \psi(y)\,dy. \tag{3.3.18}$$

For any a and b, $1 \le a < b < \infty$,

$$\liminf_{\lambda \downarrow 0} \int_a^b g_\lambda(y)\,dy \ge \sigma\psi(b)(b-a).$$

Therefore, for any $M > 1$

$$\liminf_{\lambda \downarrow 0} \int_1^M g_\lambda(y)\,dy \ge \sigma \int_1^M \psi(y)\,dy.$$

Thus,

$$\liminf_{\lambda \downarrow 0} \int_1^{\infty} g_\lambda(y)\,dy \ge \sigma \int_1^M \psi(y)\,dy,$$

and because $M > 1$ is arbitrary, we conclude that

$$\liminf_{\lambda \downarrow 0} \int_1^{\infty} g_\lambda(y)\,dy \ge \sigma \int_1^{\infty} \psi(y)\,dy. \tag{3.3.19}$$

Combining bounds (3.3.18) and (3.3.19), we arrive at

$$\int_1^\infty g_\lambda(y)\,dy \to \sigma \int_1^\infty \psi(y)\,dy, \qquad \lambda \downarrow 0. \tag{3.3.20}$$

From (3.3.16), (3.3.17), and (3.3.20) it follows that

$$\sum_{n>1/\lambda,\, n\in A} \frac{\exp(-n\lambda)}{n} \to \sigma \int_1^\infty \frac{\exp(-y)}{y}\,dy, \qquad \lambda \downarrow 0. \tag{3.3.21}$$

From (3.3.15) and (3.3.21) we arrive at (3.3.11) with

$$\gamma = \int_0^1 \frac{1-\exp(-y)}{y}\,dy - \int_1^\infty \frac{\exp(-y)}{y}\,dy.$$

It is not difficult to see that γ is the Euler constant. By setting $A = \mathbf{N}$ in (3.3.11), we obtain the relation

$$\ln(1-\exp(-\lambda))^{-1} = \sum_{n\le 1/\lambda} 1/n - \gamma + o(1),$$

which implies that γ is the Euler constant indeed. From (3.3.11) it follows that for any fixed $\mu > 0$

$$\begin{aligned} f(\exp(-\mu/n)) &= \sum_{m\in A(n/\mu)} 1/m - \sigma\gamma + o(1) \\ &= \sum_{m\in A(n)} 1/m - \sigma \ln\mu - \sigma\gamma + o(1), \qquad n\to\infty. \end{aligned}$$

It is easily seen indeed that for $\mu > 1$

$$\begin{aligned} \sum_{m>[n/\mu],\, m\in A(n)} 1/m &= \int_{1/\mu}^1 h_n(y)\,dy + o(1) \\ &= \sigma \int_{1/\mu}^1 \frac{dy}{y} + o(1) = \sigma\ln\mu + o(1), \qquad n\to\infty, \end{aligned}$$

where

$$h_n(y) = \begin{cases} 1/y, & y\in A'(n)/n,\ y \ge [n/\mu]/n, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu < 1$

$$\sum_{m>n,\, m\in A(n/\mu)} 1/m = \sigma \int_1^{1/\mu} \frac{dy}{y} + o(1) = -\sigma\ln\mu + o(1), \qquad n\to\infty.$$

The lemma is thus proved. □

PROOF OF LEMMA 3.3.2. We fix $m \in \mathbf{N}$ and set

$$t = h(m), \quad a_n = a(m, n), \quad M(n) = \{i : i \in \mathbf{N} \cup \{0\},\ n - i \in B\}.$$

For $l > n$ and $k = l - n$,

$$\begin{aligned}\sum_{j=0}^{l} a_j(\chi_B(l-j) - \chi_B(n-j)) &= \sum_{j \in M(l)} a_j - \sum_{j \in M(n)} a_j \\ &= \sum_{j \in M(l),\ j<k} a_j + \sum_{j \in M(n)} (a_{j+k} - a_j).\end{aligned}$$

So, from (3.2.23) it follows that there exists $\rho \in [-1, 1]$ such that

$$a_l - a_n = \frac{\tau(t-1)}{n} S_l + \frac{\theta(l-n)}{n^2} S_l + \frac{\rho}{n} S_k - \frac{t}{n} \sum_{j \in M(n)} (a_{j+k} - a_j). \tag{3.3.22}$$

In the rest of the proof, we assume that $m \asymp n$ as $n \to \infty$. From (3.3.22) and Theorem 1.5.7 it follows that

$$a_n = O(R(n)), \qquad n \to \infty, \tag{3.3.23}$$

where

$$R(n) = r(n) n^{-1-\alpha} = n^{\sigma-1} L_0(n).$$

By (3.3.22) and (3.3.23), as $j \to \infty$ and $k = o(j)$ we obtain

$$a_{j+k} - a_j = -\frac{t}{j} \sum_{i \in M(j)} (a_{i+k} - a_i) + o(R(j)). \tag{3.3.24}$$

We fix $\varepsilon \in (0, 1)$ and set

$$M(n, \varepsilon) = \{j : j \geq n\varepsilon, n - j \in B\}.$$

Substituting (3.3.24) into (3.3.22), as $n \to \infty$ and $k = o(n)$ we obtain

$$\begin{aligned}a_l - a_n &= o\left(\frac{S_n}{n}\right) + O\left(\frac{S_{n\varepsilon}}{n}\right) + \frac{t^2}{n} \sum_{j \in M(n,\varepsilon)} \frac{1}{j} \left(\sum_{i \in M(j)} (a_{i+k} - a_i) + o(R(j)) \right) \\ &= O\left(\frac{S_{n\varepsilon}}{n}\right) + o(R(n)) + \frac{t^2}{n} \sum_{j \in M(n,\varepsilon)} \frac{1}{j} \sum_{i \in M(j)} (a_{i+k} - a_i).\end{aligned} \tag{3.3.25}$$

Since $t \to 1$ as $m \to \infty$, from (3.3.23) and (3.3.25) it follows that for a constant $C < \infty$ the formula

$$\limsup_{n\to\infty,\ k=o(n)} \frac{|a_l - a_n|}{R(n)} \leq C\varepsilon^{\sigma} + \limsup_{n\to\infty,\ k=o(n)} \frac{|G(n,k,\varepsilon)|}{nR(n)}$$

holds true, where

$$G(n,k,\varepsilon) = \sum_{j\in M(n,\varepsilon)} \frac{1}{j} \sum_{i\in M(j)} (a_{i+k} - a_i).$$

In other words,

$$\limsup_{n\to\infty,\ k=o(n)} \frac{|a_l - a_n|}{R(n)} \le \limsup_{\varepsilon\to 0} \limsup_{n\to\infty,\ k=o(n)} \frac{|G(n,k,\varepsilon)|}{nR(n)}. \tag{3.3.26}$$

Changing the summation order in $G(n,k,\varepsilon)$, we arrive at the equality

$$G(n,k,\varepsilon) = \sum_{i=0}^{n} a_{i+k} l(i,n) - \sum_{i=0}^{n} a_i l(i,n),$$

where

$$l(i,n) = \sum_{j\ge n\varepsilon,\ j-i,n-j\in B} 1/j.$$

Thus,

$$\begin{aligned} G(n,k,\varepsilon) &= \sum_{i=k}^{l} a_i l(i-k,n) - \sum_{i=0}^{n} a_i l(i,n) \\ &= \sum_{i=k}^{n} a_i (l(i-k,n) - l(i,n)) + \sum_{i=n+1}^{l} a_i l(i-k,n) - \sum_{i=0}^{k} a_i l(i-k,n). \end{aligned}$$

We observe that

$$l(i,n) \le n/(n\varepsilon) = 1/\varepsilon,$$

and therefore, as $n\to\infty$ and $k = o(n)$,

$$G(n,k,\varepsilon) = \sum_{i=k}^{n} a_i (l(i-k,n) - l(i,n)) + o(nR(n)). \tag{3.3.27}$$

For $i = 0, 1, \dots, n$, the inequalities

$$l(i,n) \le \sum_{j=[n\varepsilon]}^{n} 1/j = \ln n - \ln n\varepsilon + o(1) = \ln(1/\varepsilon) + o(1), \qquad n\to\infty,$$

are true. Therefore, in view of (3.3.23), for some constant $C_1 < \infty$ we obtain

$$\begin{aligned} &\limsup_{n\to\infty,\ k=o(n)} \left| \sum_{i=k}^{k+[n\varepsilon]} a_i (l(i-k,n) - l(i,n)) \right| / (nR(n)) \\ &\qquad \le \limsup_{n\to\infty,\ k=o(n)} \frac{S_{k+[n\varepsilon]}}{nR(n)} (2\ln(1/\varepsilon) + o(1)) \le C_1 \varepsilon^{\sigma} \ln(1/\varepsilon). \end{aligned} \tag{3.3.28}$$

Furthermore, as $n \to \infty$ and $k = o(n)$ we see that

$$\sum_{i=k+[n\varepsilon]+1}^{n} a_i(l(i-k,n) - l(i,n)) = o(S_n), \tag{3.3.29}$$

because

$$l(m+k,n) - l(m,n) \to 0, \quad n \to \infty, \quad k = o(n), \quad n\varepsilon < m \le n. \tag{3.3.30}$$

For $m > n\varepsilon$ we see that

$$l(m,n) = \sum_{j \ge n\varepsilon, j-m, n-j \in B} \frac{1}{j} = \sum_{i, n-m-i \in B} \frac{1}{i+m}.$$

Let

$$L(m,n) = \sum_{i, n-i \in B} \frac{1}{i+m}.$$

Expression (3.3.30) in terms of $L(m,n)$ takes the form

$$L(m+k, n-k) - L(m,n) \to 0, \quad m \to \infty, \quad k = o(m), \quad n \le m(1-\varepsilon)). \tag{3.3.31}$$

We set

$$Z(n) = \{i : i \in B,\ n-i \in B\}.$$

From (3.3.1) and (3.3.2) it follows that the asymptotic density of the sequence of sets $Z(n)$ is $\beta = (1-\sigma)^2$, namely, for any constant $c \in (0,1]$

$$|u : u \le cn,\ u \in Z(n)|/cn \to \beta, \qquad n \to \infty. \tag{3.3.32}$$

Without loss of generality we assume that $n = \mu m + o(m)$ as $m \to \infty$ in (3.3.31), where $\mu \in [0, 1-\varepsilon]$ (from an arbitrary sequence $n = n(m)$ such that $n \le m(1-\varepsilon)$ we extract a subsequence such that $n = \mu m + o(m)$). First we consider the case $\mu = 0$. Then

$$0 \le L(m,n) \le \frac{n}{m} = o(1), \qquad m \to \infty,$$

and

$$0 \le L(m+k, n-k) \le \frac{n}{m+k} = o(1), \qquad m \to \infty,$$

and therefore, (3.3.31) holds true in this case. Next, let $\mu > 0$. Then

$$\begin{aligned} L(m,n) &= \int_{Z'(n)} \frac{dx}{x+m} + o(1) = \int_{Z'(n)/n} \frac{dy}{y + m/n} + o(1) \\ &= \int_{Z'(n)/n} \frac{dy}{\mu^{-1} + y} + o(1), \end{aligned}$$

where

$$Z'(n) = \bigcup_{i \in Z(n)} [i, i+1].$$

As for (3.3.15), with the use of (3.3.32) we find that

$$\int_{Z'(n)/n} \frac{dy}{\mu^{-1} + y} \to \beta \int_0^1 \frac{dy}{\mu^{-1} + y} = \beta \ln(1 + \mu).$$

In other words, as $m \to \infty$, $n \sim \mu m$,

$$L(m, n) \to \beta \ln(1 + \mu).$$

Since $k = o(m)$, under these conditions we obtain $n - k \sim \mu(m + k)$. Thus,

$$L(m + k, n - k) \to \beta \ln(1 + \mu).$$

From the last two relations we derive (3.3.31) for the chosen subsequence of $n = n(m)$. Since the limit in (3.3.31) does not depend on the choice of the subsequence of $n = n(m)$, (3.3.31) is indeed true as $m \to \infty$, $k = o(m)$, uniformly in $n \le m(1 - \varepsilon)$. So (3.3.29) is proved. From (3.3.26), (3.3.27), (3.3.28), and (3.3.29) it follows that

$$a_l - a_n = o(R(n)) = o(S_n/n)$$

as $n \to \infty$, $k = o(n)$, and $l = n + k$, because $S_n \asymp nR(n)$ as $n \to \infty$ (which is proved in the same way as relation (3.2.28)). The proof of the lemma is thus complete. □

3.4. Uniformly distributed sequences

This section is of technical nature. Here we consider a new class of uniformly distributed sequences. We need this class in order to construct examples of sets A such that limit theorems 3.3.1, 3.3.2, and 3.3.3 are true.

Let a sequence of real numbers (x_n) for $n \in \mathbf{N}$ and a finite union Δ of segments of [0, 1] be given. A number $m \in \mathbf{N}$ enters into the set A if and only if $\{x_m\} \in \Delta$, where $\{a\}$ is the fractional part of the number a. Appropriately choosing (x_n) and Δ, we get any set A we wish. The objective of this section and the succeeding one consists of isolation of those sequences (x_n) for which the corresponding set A obeys relations (3.3.1) and (3.3.2) for any Δ, so limit theorems 3.3.1, 3.3.2 and 3.3.3 are true for it.

We give a well-known definition of an *uniformly distributed sequence* (Kuipers, Niederreiter, 1974, Section 1.1)).

DEFINITION 3.4.1. A sequence (x_n) is said to be *uniformly distributed modulo* 1 if for any segment $\Delta \subset [0, 1]$ as $n \to \infty$

$$|k : k \le n, \{x_k\} \in \Delta| / n \to |\Delta|,$$

where $|\Delta|$ is the Lebesgue measure of Δ.

To study the sets A, we have to introduce the following class of uniformly distributed modulo 1 sequences (Yakymiv, 1993b).

DEFINITION 3.4.2. A sequence (x_n) is *strongly uniformly distributed modulo* 1 if

$$|k : k \le n,\ \{x_k\} \in \Delta_1,\ \{x_{m-k}\} \in \Delta_2|/n \to |\Delta_1||\Delta_2| \tag{3.4.1}$$

for any segments $\Delta_1, \Delta_2 \subset [0, 1]$ and any constant $1 < C < \infty$ as $n \to \infty$ uniformly in $m \in [n, Cn]$.

In what follows, the words 'modulo 1' will be omitted for the sake of brevity.

REMARK 3.4.1. As in Remark 3.3.1, it is sufficient to assume that relation (3.4.1) holds for an arbitrary sequence $m = m(n)$ such that $m \ge n$ and $m = O(n)$ as $n \to \infty$.

REMARK 3.4.2. If a sequence (x_n) is strongly uniformly distributed, then the corresponding set A obeys relations (3.3.1) and (3.3.2) with $\sigma = |\Delta|$.

So, in what follows we look for strongly uniformly distributed sequences. It is easy to see that any strongly uniformly distributed sequence is uniformly distributed ($\Delta_1 = \Delta$, $\Delta_2 = [0, 1]$ in Definition 3.4.2). The reverse is, generally speaking, not true. The counterexample is as follows: $x_n = \theta n$, where θ is an irrational number.

For the strongly uniformly distributed sequences, the following analogue of the well-known Weyl's criterion for uniformly distributed sequences ((Kuipers, Niederreiter, 1974, Section 1.2)).

THEOREM 3.4.1. *A sequence (x_n) is strongly uniformly distributed if and only if*

$$\sum_{k=1}^{n} \exp(2\pi i (a x_k + b x_{m-k}))/n \to 0 \tag{3.4.2}$$

uniformly in $m \in [n, Cn]$ for any integers a, b, $(a, b) \ne \varnothing$, and any constant $1 < C < \infty$ as $n \to \infty$.

In our counterexample, (3.4.2) is broken for $a = b$. In order to prove Theorem 3.4.1, we follow the pattern used in (Kuipers, Niederreiter, 1974, Section 1.6) to validate Weyl's criterion for multidimensional uniformly distributed sequences.

Let

$$I = \{(x, y) \in \mathbf{R}^2,\ 0 \le x \le 1,\ 0 \le y \le 1\}.$$

For any set $J \subset \mathbf{R}^2$, we set

$$\chi_J(x, y) = \begin{cases} 1, & (x, y) \in J, \\ 0, & (x, y) \notin J. \end{cases}$$

THEOREM 3.4.2. *A sequence (x_n) is strongly uniformly distributed if and only if for any real-valued continuous function $f(x, y)$ on I and any constant $1 < C < \infty$*

$$\frac{1}{n} \sum_{k=1}^{n} f(\{x_k\}, \{x_{m-k}\}) \to \int_I f(x, y)\, dx\, dy \tag{3.4.3}$$

as $n \to \infty$ uniformly in $m \in [n, Cn]$.

PROOF. Let a sequence (x_n) be strongly uniformly distributed. Then for any rectangle $J = \Delta_1 \times \Delta_2$, where Δ_1, Δ_2 are segments in [0, 1], and for any constant $1 < C < \infty$

$$\frac{1}{n}\sum_{k=1}^{n} \chi_J(\{x_k\}, \{x_{m-k}\}) \to \int_I \chi_J(x, y)\, dx\, dy$$

as $n \to \infty$ uniformly in $m \in [n, Cn]$. Therefore, for an arbitrary step function

$$f(x, y) = \sum_{k=1}^{M} c_k \chi_{J_k}$$

where J_k are arbitrary rectangles in I, as $n \to \infty$ we obtain

$$\frac{1}{n}\sum_{k=1}^{n} f(\{x_k\}, \{x_{m-k}\}) \to \int_I f(x, y)\, dx\, dy$$

uniformly in $m \in [n, Cn]$. Further, let $f(x, y)$ be continuous on I. From the definition of the two-dimensional Riemann integral it follows that for an arbitrary $\varepsilon > 0$ there exist step functions f_1 and f_2 such that

$$f_1(x, y) \le f(x, y) \le f_2(x, y)$$

for $(x, y) \in I$, and

$$\int_I (f_2(x, y) - f_1(x, y))\, dx \le \varepsilon.$$

We choose an arbitrary sequence $m = m(n)$ such that $m \ge n$, $m = O(n)$ as $n \to \infty$. Then

$$\begin{aligned}
\int_I f(x, y)\, dx\, dy - \varepsilon \le \int_I f_1(x, y)\, dx\, dy &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_1(\{x_k\}, \{x_{m-k}\}) \\
&\le \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\{x_k\}, \{x_{m-k}\}) \le \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\{x_k\}, \{x_{m-k}\}) \\
&\le \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_2(\{x_k\}, \{x_{m-k}\}) = \int_I f_2(x, y)\, dx\, dy \\
&\le \int_I f(x, y)\, dx\, dy + \varepsilon.
\end{aligned}$$

Since ε is chosen arbitrarily, there exists

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\{x_k\}, \{x_{m-k}\}) = \int_I f(x, y)\, dx\, dy.$$

Since the last relation holds for any sequence $m = m(n)$ which possess the stated above properties, we conclude that (3.4.3) holds uniformly in $m \in [n, Cn]$ for any fixed constant $1 < C < \infty$.

Vice versa, let (3.4.3) be true and $J = \Delta_1 \times \Delta_2$, where Δ_1, Δ_2 are arbitrary segments in [0, 1]. For an arbitrary $\varepsilon > 0$ there exist continuous functions g_1 and g_2 such that

$$g_1(x, y) \le \chi_J(x, y) \le g_2(x, y), \qquad \forall (x, y) \in I,$$

and

$$\int_I (g_2(x, y) - g_1(x, y))\, dx\, dy < \varepsilon.$$

We choose an arbitrary sequence $m = m(n)$ such that $m \ge n$, $m = O(n)$ as $n \to \infty$. Then

$$\begin{aligned} |J| - \varepsilon &\le \int_I g_2(x, y)\, dx\, dy - \varepsilon < \int_I g_1(x, y)\, dx\, dy \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_1(\{x_k\}, \{x_{m-k}\}) \le \varliminf_{n\to\infty} \frac{1}{n} |k : k \le n,\ (\{x_k\}, \{x_{m-k}\}) \in J| \\ &\le \varlimsup_{n\to\infty} \frac{1}{n} |k : k \le n,\ (\{x_k\}, \{x_{m-k}\}) \in J| \le \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_2(\{x_k\}, \{x_{m-k}\}) \\ &= \int_I g_2(x, y)\, dx\, dy < \int_I g_1(x, y)\, dx\, dy + \varepsilon \le |J| + \varepsilon. \end{aligned}$$

Because of arbitrariness of ε, there exists

$$\lim_{n\to\infty} \frac{1}{n} |k : k \le n,\ (\{x_k\}, \{x_{m-k}\}) \in J| = |J|.$$

Since the last relation holds true for an arbitrary sequence $m = m(n)$ which possesses the stated above properties, we conclude that (3.4.1) holds uniformly in $m \in [n, Cn]$ for any fixed constant $1 < C < \infty$. The theorem is thus proved. □

COROLLARY 3.4.1. *A sequence (x_n) is strongly uniformly distributed if and only if* (3.4.3) *holds true for an arbitrary complex-valued continuous function $f(x, y)$ on I.*

COROLLARY 3.4.2. *A sequence (x_n) is strongly uniformly distributed if and only if*

$$\frac{1}{n} \sum_{k=1}^{n} f(x_k, x_{m-k}) \to \int_I f(x, y)\, dx\, dy \tag{3.4.4}$$

uniformly in $m \in [n, Cn]$ for an arbitrary complex-valued continuous function $f(x, y)$ on $\mathbf{R}^2$ which is periodic in x and y with period 1, *for an arbitrary constant $C \in [1, \infty)$.*

PROOF OF COROLLARY 3.4.2. The necessity follows from Corollary 3.4.1. The proof of the sufficiency repeats that for Theorem 3.4.2 with the exception that the functions g_1 and g_2 must satisfy the additional constraints

$$g_i(x, 0) = g_i(x, 1), \quad g_i(0, y) = g_i(1, y) \quad \forall (x, y) \in I, \quad i = 1, 2;$$

so that (3.4.4) can be applied to the periodic continuations of the functions g_1 and g_2 to $\mathbf{R}^2$. □

PROOF OF THEOREM 3.4.1. The necessity follows from Corollary 3.4.2. Let us prove the sufficiency. The set of all finite linear combinations of the functions of the form $\exp(2\pi i(ax+by))$, where a, b are integers, with complex coefficients is dense (with respect to the uniform norm) in the space of all continuous complex-valued functions $f(x, y)$ with period 1 in each variable (Kuipers, Niederreiter, 1974). We choose an arbitrary complex-valued continuous function $f(x, y)$ in $\mathbf{R}^2$ which is periodic in x and y with period 1. By the abovesaid, for any $\varepsilon > 0$ there exist $M \in \mathbf{N}$, complex numbers $c_1, \dots, c_M$, and integers $a_1, \dots, a_M, b_1, \dots, b_M$ such that

$$|f(x, y) - \psi(x, y)| < \varepsilon, \qquad \forall (x, y) \in \mathbf{R}^2, \tag{3.4.5}$$

where

$$\psi(x, y) = \sum_{l=1}^{M} c_l \exp(2\pi i(xa_l + yb_l)), \qquad (x, y) \in \mathbf{R}^2.$$

We choose an arbitrary sequence $m = m(n)$ such that $m \geq n$, $m = O(n)$, $n \to \infty$. The inequality

$$\begin{aligned}\left|\int_I f(x, y)\,dx\,dy - \frac{1}{n}\sum_{k=1}^{n} f(x_k, x_{m-k})\right| &\leq \left|\int_I (f(x, y) - \psi(x, y))\,dx\,dy\right| \\ &+ \left|\int_I \psi(x, y)\,dx\,dy - \frac{1}{n}\sum_{k=1}^{n} \psi(x_k, x_{m-k})\right| \\ &+ \left|\frac{1}{n}\sum_{k=1}^{n}(f(x_k, x_{m-k}) - \psi(x_k, x_{m-k}))\right|\end{aligned}$$

is true. The first and third terms in the last inequality are smaller than ε by (3.4.5), and the second term is smaller than ε for sufficiently large n due to (3.4.1). In order to prove the theorem, it remains to make use of Corollary 3.4.2. □

To close this section, we give some necessary information from the monograph (Kuipers, Niederreiter, 1974).

The two lemmas below are Lemmas 2.1 and 2.2 in (Kuipers, Niederreiter, 1974, Section 1.2) respectively.

LEMMA 3.4.1. *We assume that a real-valued function f possesses a monotone derivative f' on $[a, b]$, while either $f'(x) \geq \lambda > 0$ or $f'(x) \leq -\lambda < 0$ for $x \in [a, b]$. We set*

$$J = \int_a^b \exp(2\pi i f(x))\,dx.$$

Then

$$|J| \leq \frac{1}{\lambda}.$$

LEMMA 3.4.2. *Let f be twice differentiable on $[a,b]$ and either $f''(x) \geq \rho > 0$ or $f''(x) \leq -\rho < 0$ for $x \in [a,b]$. Then the integral J in Lemma 3.4.1 obeys the inequality*

$$|J| < \frac{4}{\sqrt{\rho}}.$$

The theorem below is Theorem 2.7 in (Kuipers, Niederreiter, 1974, Section 1.2).

THEOREM 3.4.3. *Let a and b be some integers, and let f be twice differentiable on $[a,b]$, while either $f''(x) \geq \rho > 0$ or $f''(x) \leq -\rho < 0$ for $x \in [a,b]$. Then*

$$\left|\sum_{n=a}^{b} \exp(2\pi i f(n))\right| \leq (|f'(b) - f'(a)| + 2)\left(\frac{4}{\sqrt{\rho}} + 3\right).$$

The lemma below is Lemma 3.1 in (Kuipers, Niederreiter, 1974, Section 1.3).

LEMMA 3.4.3 (van der Corput fundamental inequality). *Let $u_1, \dots, u_n$ be some complex numbers, and let H be an integer, $1 \leq H \leq n$. Then*

$$H^2\left|\sum_{t=1}^{n} u_t\right|^2 \leq H(n+H-1)\sum_{t=1}^{n}|u_n|^2 + 2(n+H-1)\sum_{h=1}^{H-1}(H-h)\Re\sum_{t=1}^{n-h} u_t\bar{u}_{t+h}.$$

3.5. Examples of sets A

Let a real valued function $g(t)$ be given for $t > 0$, and let a finite union Δ of segments of $[0,1]$ be given. A number $m \in \mathbf{N}$ enters into the set A if and only if $\{g(m)\} \in \Delta$. As we have seen in the preceding section, if the sequence $(g(n))$ is strongly uniformly distributed, then for any Δ the corresponding set A obeys relations (3.3.1) and (3.3.2), so limit theorems 3.3.1, 3.3.2 and 3.3.3 are valid. We thus turn to constructing strongly uniformly distributed sequences.

The simplest class of strongly uniformly distributed sequences $(g(n))$ is described in the following theorem.

THEOREM 3.5.1. *Let $g'(t) \geq 0$, $g'(t)$ be convex, and let $g'(t)$ monotonically tend to zero as $t \to \infty$. If $tg'(t)$ and $t^2|g''(t)|$ tend to infinity as $t \to \infty$, then $(g(n))$ is strongly uniformly distributed.*

PROOF. Let

$$J = \int_0^n \exp(2\pi i(ag(x) + bg(m-x)))\,dx,$$

where a and b are integers, $(a,b) \neq 0$. Since $g'(t) \to 0$ as $t \to \infty$, we see that

$$\sum_{t=1}^{n} \exp(2\pi i(ag(t) + bg(m-t))) = J + o(n) \tag{3.5.1}$$

as $n \to \infty$ for an arbitrary sequence $m = m(n)$ such that $m \geq n$ and $m = O(n)$ as $n \to \infty$. Let $a \neq 0$. We set $\theta = |b/a|$. Let $b/a \leq 0$. Then by virtue of Lemma 3.4.1

$$|J| \leq \frac{2}{|a|\min\limits_{t\leq n}(g'(t) + \theta g'(m-t))} \leq \frac{2}{|a|g'(n)}. \tag{3.5.2}$$

We observe that $m = m(n)$ is a sequence such that $m \geq n$ and $m = O(n)$ as $n \to \infty$. So, from (3.5.1), (3.5.2), and the fact that $tg'(t) \to \infty$ as $t \to \infty$ it follows that the sequence $x_n = g(n)$ obeys relation (3.4.2). Let $b/a > 0$. Then Lemma 3.4.2 yields the inequality

$$|J| \leq \frac{4}{\sqrt{|a| \min_{t \leq n} |g''(t) + \theta g''(m-t)|}} \leq \frac{4}{\sqrt{|a| g''(n)}}. \tag{3.5.3}$$

From (3.5.1), (3.5.3) and the fact that $t^2 |g''(t)| \to \infty$ as $t \to \infty$) we arrive at (3.4.2). Finally, in the case where $a = 0$, by Lemma 3.4.1 we obtain the inequality

$$|J| \leq 2/|b| g'(m).$$

From the last inequality and relation (3.5.1) in the same way as before we find that (3.4.2) holds. Thus, the sequence $x_n = g(n)$ satisfies hypothesis (3.4.2) of Theorem 3.4.1, which states that the sequence $(g(n))$ is strongly uniformly distributed indeed. □

COROLLARY 3.5.1. *Let*

$$g(t) = t^\alpha l(t), \qquad t > 0, \tag{3.5.4}$$

where $\alpha \in (0, 1)$*, and let the function* $l(t)$ *be slowly varying at infinity, while*

$$\frac{d^n}{dt^n} l(t) = o(t^{-n} l(t)) \tag{3.5.5}$$

for $n = 1, 2, 3$ *as* $t \to \infty$*. Then the sequence* $(g(n))$ *is strongly uniformly distributed.*

From (3.5.4) and (3.5.5) it indeed follows that

$$g'(t) = \alpha t^{\alpha-1} l(t) + t^\alpha l'(t) \sim \alpha t^{\alpha-1} l(t), \tag{3.5.6}$$

$$g''(t) = \alpha(\alpha-1) t^{\alpha-2} l(t) + 2\alpha t^{\alpha-1} l'(t) + t^\alpha l''(t) \sim \alpha(\alpha-1) t^{\alpha-2} l(t), \tag{3.5.7}$$

$$\begin{aligned} g'''(t) &= \alpha(\alpha-1)(\alpha-2) t^{\alpha-3} l(t) + 3\alpha(\alpha-1) t^{\alpha-2} l'(t) + 3\alpha t^{\alpha-1} l''(t) + t^\alpha l'''(t) \\ &\sim \alpha(\alpha-1)(\alpha-2) t^{\alpha-3} l(t) \end{aligned} \tag{3.5.8}$$

as $t \to \infty$. From (3.5.6), (3.5.7), and (3.5.8) it follows that the hypotheses of Theorem 3.5.1 are fulfilled.

REMARK 3.5.1. For any slowly varying at infinity function, there exists a function $l(t)$ which is equivalent to it at infinity and obeys (3.5.5) for all $n \in \mathbf{N}$ (see Theorem 1.1.3). As examples of functions $l(t)$ which obey (3.5.5), we mention the functions $\ln t$, $\ln \ln t$, $\exp(\ln^\beta t)$ with $\beta < 1$, their powers and products.

In what follows we consider the cases where equality (3.5.4) holds for some slowly varying at infinity function $l(t)$ and $\alpha \geq 1$; the cases of integer and non-integer α are studied separately.

THEOREM 3.5.2. *For some non-integer* $\alpha > 1$ *and a slowly varying at infinity function* $l(t)$*, let equality* (3.5.4) *hold and relation* (3.5.5) *be true for all* $n = 1, 2, \ldots, [\alpha] + 2$*, where* $[\alpha]$ *stands for the integer part of* α*. Then the sequence* $(g(n))$ *is strongly uniformly distributed.*

For $m \in \mathbf{N}$, $h_1, \dots, h_m \in \mathbf{N}$ we set

$$g(h_1, x) = g(x + h_1) - g(x), \quad \dots,$$
$$g(h_1, \dots, h_m, x) = g(h_1, \dots, h_{m-1}, x + h_m) - g(h_1, \dots, h_{m-1}, x).$$

We formulate a series of auxiliary assertions.

LEMMA 3.5.1. *If the sequence $(g(h_1, \dots, h_k, n))$ is strongly uniformly distributed for some $k \in \mathbf{N}$ and arbitrary $h_1, \dots, h_k \in \mathbf{N}$, then so is the sequence $(g(n))$.*

LEMMA 3.5.2. *Let the hypotheses of Theorem 3.5.2 be fulfilled. Then*

$$\frac{d^m}{dt^m} g(t) \sim \alpha^{[m]} t^{\alpha - m} l(t) \tag{3.5.9}$$

for arbitrary $m = 1, 2, \dots, [\alpha] + 2$ as $t \to \infty$, where $\alpha^{[m]} = \alpha(\alpha - 1) \cdots (\alpha - m + 1)$.

LEMMA 3.5.3. *We fix $h_1, \dots, h_m \in \mathbf{N}$, where $m = [\alpha] - 1$, and set $g_m(t) = g(h_1, \dots, h_m, t)$. If the hypotheses of Theorem 3.5.2 are fulfilled, then, as $t \to \infty$,*

$$g'_m(t) \sim h\alpha^{[m+1]} t^{\beta} l(t), \tag{3.5.10}$$
$$g''_m(t) \sim h\alpha^{[m+2]} t^{\beta-1} l(t), \tag{3.5.11}$$
$$g'''_m(t) \sim h\alpha^{[m+3]} t^{\beta-2} l(t), \tag{3.5.12}$$

where $\beta = \{\alpha\}$, $h = h_1 \dots h_m$.

LEMMA 3.5.4. *For some function $g(t)$ and a slowly varying at infinity function $l(t)$, let*

$$g'(t) \sim t^{\beta} l(t), \tag{3.5.13}$$
$$g''(t) \sim \beta t^{\beta-1} l(t), \tag{3.5.14}$$
$$g'''(t) \sim \beta(\beta - 1) t^{\beta-2} l(t) \tag{3.5.15}$$

for $\beta \in (0, 1)$ as $t \to \infty$. Then the sequence $(g(n))$ is strongly uniformly distributed.

Before proving Theorem 3.5.2, we validate Lemmas 3.5.1–3.5.4.

PROOF OF LEMMA 3.5.1. First we prove the lemma for $k = 1$. Let the sequence $(g(h, n))$ be strongly uniformly distributed for an arbitrary $h \in \mathbf{N}$. By the van der Corput fundamental inequality (Lemma 3.4.3), for any $n, H, m \in \mathbf{N}$, $m \geq n$, and integers a, b, $(a, b) \neq 0$,

$$H^2 \left| \sum_{t=1}^{n} \exp(2\pi i (ag(t) + bg(m - t))) \right|^2 \leq H(n + H - 1)n$$
$$+ 2(n + H - 1) \sum_{h=1}^{H-1} (H - h) \Re \sum_{t=1}^{n-h} \exp(2\pi i (ag(t) + bg(m - t) - ag(t + h) - bg(m - t - h))).$$

By dividing both sides of the inequality by n^2H^2, hence we obtain

$$\left|\frac{1}{n}\sum_{t=1}^{n}\exp(2\pi i(ag(t)+bg(m-t)))\right|^2 \le \frac{(n+H-1)}{nH} + 2\sum_{h=1}^{H-1}\frac{(n+H-1)(H-h)}{n^2H^2}\left|\sum_{t=1}^{n-h}\exp(2\pi i(ag(h,t)+bg(h,m-t)))\right|.$$

Thus, for an arbitrary sequence $m=m(n)$ such that $m \ge n$ and $m = O(n)$ as $n \to \infty$ we see that

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{t=1}^{n}\exp(2\pi i(ag(t)+bg(m-t)))\right|^2 \le \frac{1}{H} + 2\sum_{h=1}^{H-1}\frac{H-h}{H^2}\lim_{n\to\infty}\left|\frac{1}{n}\sum_{t=1}^{n-h}\exp(2\pi i(ag(h,t)+bg(h,m-t)))\right| = \frac{1}{H}$$

by virtue of Theorem 3.4.1. Since H is arbitrarily chosen, from the last inequality by virtue of Theorem 3.4.1 we conclude that the sequence $(g(n))$ is strongly uniformly distributed.

Now let $k > 1$, and let the lemma be true for $k-1$. If the sequence $(g(h_1,\dots,h_k,n))$ is strongly uniformly distributed for any $h_1,\dots,h_k \in \mathbf{N}$, then by the just proved the sequence $(g(h_1,\dots,h_{k-1},n))$ is strongly uniformly distributed, which by the induction assumption implies that the sequence $(g(n))$ is strongly uniformly distributed. The lemma is proved. □

PROOF OF LEMMA 3.5.2. By (3.5.4) and (3.5.5), as $t \to \infty$

$$\frac{d}{dt}g(t) = \frac{d}{dt}(t^\alpha l(t)) = \alpha t^{\alpha-1}l(t) + t^\alpha l'(t) = \alpha t^{\alpha-1}l(t) + t^\alpha o(t^{-1}l(t)) \sim \alpha t^{\alpha-1}l(t).$$

If $m > 1$, $m \le [\alpha]+2$, and the lemma is true for $1,\dots,m-1$, then by (3.5.5) as $t \to \infty$

$$\frac{d^m}{dt^m}g(t) = \sum_{n=0}^{m}\binom{m}{n}\alpha^{[n]}t^{\alpha-n}l^{(m-n)}(t) = \alpha^{[m]}t^{\alpha-m}l(t) + \sum_{n=0}^{m}\binom{m}{n}\alpha^{[n]}t^{\alpha-n}o(t^{-m+n}l(t)) \sim \alpha^{[m]}t^{\alpha-m}l(t).$$

The lemma is proved. □

PROOF OF LEMMA 3.5.3. It is easily seen that for any $k=1,\dots,m$

$$g_k(t) = \int_0^{h_1}\cdots\int_0^{h_k} g^{(k)}(t+x_1+\dots+x_k)\,dx_1\cdots dx_k, \tag{3.5.16}$$

because

$$g_1(t) = g(h_1, t) = g(t + h_1) - g(t) = \int_0^{h_1} g'(t + x_1)\, dx_1;$$

if for $k > 1, k \le m$

$$g_{k-1}(t) = \int_0^{h_1} \cdots \int_0^{h_{k-1}} g^{(k-1)}(t + x_1 + \cdots + x_{k-1})\, dx_1 \cdots dx_{k-1},$$

then

$$\begin{aligned} g_k(t) &= g_{k-1}(t + h_k) - g_{k-1}(t) \\ &= \int_0^{h_1} \cdots \int_0^{h_{k-1}} (g^{(k-1)}(t + h_k + x_1 + \cdots + x_{k-1}) \\ &\quad - g^{(k-1)}(t + x_1 + \cdots + x_{k-1}))\, dx_1 \cdots dx_{k-1} \\ &= \int_0^{h_1} \cdots \int_0^{h_k} g^{(k)}(t + x_1 + \cdots + x_k)\, dx_1 \cdots dx_k, \end{aligned}$$

which implies (3.5.16). Therefore,

$$g_m(t) = \int_0^{h_1} \cdots \int_0^{h_m} g^{(m)}(t + x_1 + \cdots + x_m)\, dx_1 \cdots dx_m = hg^{(m)}(t + t_1)$$

for some $t_1 \in [0, h_1 + \cdots + h_m]$. So, from Lemma 3.5.2 it now follows that, as $t \to \infty$,

$$g_m'(t) \sim h\alpha^{[m+1]}(t + t_1)^{\beta} l(t + t_1) \sim h\alpha^{[m+1]} t^{\beta} l(t).$$

Similarly, for some $t_2, t_3 \in [0, h_1 + \cdots + h_m]$ we find that

$$\begin{aligned} g_m''(t) &\sim h\alpha^{[m+2]}(t + t_2)^{\beta-1} l(t + t_2) \sim h\alpha^{[m+2]} t^{\beta-1} l(t), \\ g_m'''(t) &\sim h\alpha^{[m+3]}(t + t_3)^{\beta-2} l(t + t_3) \sim h\alpha^{[m+3]} t^{\beta-2} l(t) \end{aligned}$$

as $t \to \infty$. The lemma is thus proved. □

PROOF OF LEMMA 3.5.4. Making use of Theorem 3.4.1, we check whether relation (3.4.2) holds or not for the sequence $x_n = g(n)$. We assume that $a \ne 0$ and set $\theta = |b/a|$, $f(t) = ag(t) + bg(m - t)$. Let $m = m(n) \in \mathbf{N}$ be some sequence obeying the conditions $m \ge n$ and $m = O(n)$ as $n \to \infty$. We first consider the case where $b/a \ge 0$. By (3.5.13), as $n \to \infty, t \le n, t \sim n$ we obtain

$$f'(t) = a(g'(t) - \theta g'(m - t)) = O(n^{\beta} l(n)). \tag{3.5.17}$$

By virtue of (3.5.14) and (3.5.15) there exists a positive integer M such that the function $g''(t)$ is positive and monotonically decreases for $t \ge M$. Then

$$|f''(t)| = |a|(g''(t) + \theta g''(m - t)) \ge |a| g''(n) \asymp n^{\beta-1} l(n) \tag{3.5.18}$$

for $n \geq 2M$, $t \in [M, n-M]$. From relations (3.5.17), (3.5.18), and Theorem 3.4.3 it follows that

$$\sum_{t=M}^{n-M} \exp(2\pi i f(t)) \leq (|f'(n-M) - f'(M)| + 2)\left(\frac{4}{\sqrt{\rho}+3}\right)$$

$$= O(n^{\beta} l(n)) O\left(\frac{1}{\sqrt{n^{\beta-1} l(n)}}\right) = O(n^{(1+\beta)/2}\sqrt{l(n)}) = o(n),$$

as $n \to \infty$, $m = m(n) \geq n$, and $m = O(n)$, where

$$\rho = \inf_{t \in [M, n-M]} |f''(t)|,$$

because $\beta < 1$. As $n \to \infty$, we see that

$$\sum_{t=1}^{M} \exp(2\pi i f(t)) + \sum_{t=n-M}^{n} \exp(2\pi i f(t)) = O(1).$$

Thus, relation (3.4.2) for the sequence $x_n = g(n)$ is true in the case where $b/a \geq 0$. Now let $b/a < 0$. For the sake of definiteness, let $a > 0$. By virtue of (3.5.15), there exists M_1 such that

$$f'''(t)/a = g'''(t) + \theta g'''(m-t) < 0 \tag{3.5.19}$$

for $m \geq 2M_1$ and $t \in [M_1, m - M_1]$. In view of (3.5.14), we can also consider the relations

$$f''(M_1)/a = g''(M_1) - \theta g''(m - M_1) > 0,$$
$$f''(m - M_1)/a = g''(m - M_1) - \theta g''(M_1) < 0$$

as being true for $m \geq M_2$, where $M_2 > 2M_1$ is some positive integer. Since $f''(t)/a$ decreases for $t \in [M_1, m - M_1]$ (see (3.5.19)), we see that for $m \geq M_2$ there exists a unique root τ of the equation $f''(\tau) = 0$ in $[M_1, m - M_1]$. It is clear that τ obeys the equality

$$g''(\tau) = \theta g''(m - \tau). \tag{3.5.20}$$

It is easily seen that $\tau \asymp m$, $m - \tau \asymp m$ as $m \to \infty$; if this were not the case, then (3.5.20) and the regular variation of $g''(t)$ as $t \to \infty$ would be both broken. From (3.5.20) and (3.5.14) we find that, as $m \to \infty$,

$$\tau^{\beta-1} l(\tau) \sim \theta (m-\tau)^{\beta-1} l(m-\tau).$$

Since $\tau \asymp m$, $m - \tau \asymp m$ as $m \to \infty$ and the function $l(t)$ is slowly varying at infinity, we obtain, as $m \to \infty$,

$$\tau^{\beta-1} \sim \theta (m-\tau)^{\beta-1},$$

or

$$\left(\frac{m-\tau}{\tau}\right)^{\beta-1} \to \theta^{-1},$$

which yields, as $m \to \infty$,

$$(m-\tau)/\tau \to \theta^\gamma, \qquad m/\tau \to \theta^\gamma + 1,$$

where $\gamma = (1-\beta)^{-1}$. Thus,

$$\tau \sim \frac{m}{1+\theta^\gamma}$$

as $m \to \infty$. We choose some $\delta \in (0,1)$, $\delta \le \theta^\gamma$, and let

$$t_1 = (1-\delta)m/(1+\theta^\gamma), \qquad t_2 = (1+\delta)m/(1+\theta^\gamma).$$

For some $M_3 > M_2$, let for $m \ge M_3$

$$M_1 < t_1 < \tau < t_2 < m - M_1.$$

Then by the mean value theorem there exists $\nu \in [t_1, \tau]$ such that

$$f''(t)/a \ge f''(t_1)/a = (f''(t_1) - f''(\tau))/a = -(\tau - t_1)f'''(\nu)/a$$

for $m \ge M_3$ and $t \in [M_1, t_1]$. As $m \to \infty$, we see that

$$\begin{aligned} -(\tau - t_1)f'''(\nu)/a &\sim \frac{-m\delta}{1+\theta^\gamma}(g'''(\nu) + \theta g'''(m-\nu)) \\ &\sim \frac{m\delta}{1+\theta^\gamma}\beta(1-\beta)l(m)(\nu^{\beta-2} + \theta(m-\nu)^{\beta-2}). \end{aligned}$$

Therefore, there exists $M_4 > M_3$ such that

$$f''(t)/a \ge cl(m)(\nu^{\beta-2} + \theta(m-\nu)^{\beta-2})m \ge cl(m)t_2^{\beta-2}m = c_1 m^{\beta-1}l(m) \qquad (3.5.21)$$

for $m \ge M_4$, $t \in [M_1, t_1]$, where

$$c = \frac{\delta\beta(1-\beta)}{2(1+\theta^\gamma)}, \qquad c_1 = c\left(\frac{1+\delta}{1+\theta^\gamma}\right)^{\beta-2}.$$

Furthermore, by the mean value theorem there exists some $\mu \in [\tau, t_2]$ such that

$$\begin{aligned} f''(t)/a &\le f''(t_2)/a = (f''(t_2) - f''(\tau))/a \\ &= (t_2 - \tau)f'''(\mu)/a \sim \frac{m\delta}{1+\theta^\gamma}(g'''(\mu) + \theta g'''(m-\mu)), \qquad m \to \infty, \end{aligned}$$

for $t \in [t_2, m - M_1]$. The last expression is, as $m \to \infty$,

$$-(1+o(1))\frac{m\delta}{1+\theta^\gamma}l(m)\beta(1-\beta)(\mu^{\beta-2} + \theta(m-\mu)^{\beta-2}).$$

Hence it follows that

$$\begin{aligned} f''(t)/a &\le -c_3(\mu^{\beta-2} + \theta(m-\mu)^{\beta-2})ml(m) \le -c_3 m\mu^{\beta-2}l(m) \\ &\le -c_3 m t_1^{\beta-2} l(m) = -c_4 m^{\beta-1} l(m), \qquad (3.5.22) \end{aligned}$$

for some constants $c_3 > 0$ and $M_5 > M_4$, where $t \in [t_2, m - M_1]$, $m \geq M_5$, and

$$c_4 = c_3 \left(\frac{1+\delta}{1+\theta^\gamma} \right)^{\beta-2}.$$

By virtue of Theorem 3.4.3, from relations (3.5.21) and (3.5.22) it follows that the inequality

$$\begin{aligned} &\left| \sum_{t=M_1}^{n-M_1} \exp(2\pi i f(t)) \right| \\ &\qquad \leq (|f'(M_1) + |f'(n-M_1)| + |f'(t_1)| + |f'(t_2)| + 4) \left(\frac{4}{\sqrt{c_5 m^{\beta-1} l(m)}} + 3 \right) \end{aligned}$$

holds true for $m \geq M_5$ and $n \leq m$, where $c_5 = \min(c_1, c_4)$. From (3.5.13) it follows that

$$|f'(M_1) + |f'(n-M_1)| + |f'(t_1)| + |f'(t_2)| = O(m^\beta l(m))$$

as $n \to \infty$ for an arbitrary sequence $m = m(n)$, $m = O(n)$. Therefore,

$$\begin{aligned} \sum_{t=M_1}^{n-M_1} \exp(2\pi i f(t)) &= O(m^\beta l(m)) O(m^{(1-\beta)/2} / \sqrt{l(m)}) \\ &= O(m^{(1+\beta)/2} \sqrt{l(m)}) = o(m) \end{aligned}$$

as $n \to \infty$ for $m \geq n$, $m = O(n)$. Further,

$$\left| \frac{1}{m} \sum_{t \in [t_1, t_2] \cap [1,n]} \exp(2\pi i f(t)) \right| \leq \frac{t_2 - t_1 + 1}{m} \leq \frac{2\delta}{1+\theta^\gamma} + \frac{1}{m}.$$

From the last two relations it follows that

$$\limsup_{n \to \infty} \left| \frac{1}{m} \sum_{t=1}^{n} \exp(2\pi i f(t)) \right| \leq \frac{2\delta}{1+\theta^\gamma}$$

for $m \geq n$, $m = O(n)$. Since δ, as well as the sequence $m = m(n)$ such that $m \geq n$, $m = O(n)$, are arbitrarily chosen, relation (3.4.2) holds true for the sequence $x_n = g(n)$ in the case where $b/a < 0$.

Finally, in the case where $a = 0$ relations (3.5.17), (3.5.18) are replaced by

$$\begin{aligned} f'(t) &= -bg'(m-t) = O(n^\beta l(n)), \\ |f''(t)| &= |b|g''(m-t) \geq |b|g''(m)| \asymp n^{\beta-1} l(n), \end{aligned}$$

respectively, and the rest of the proof follows the same way as in the case $b/a \geq 0$. The proof of the lemma is thus complete. □

PROOF OF THEOREM 3.5.2. Let $[\alpha] = 1$. Then the function $g(t)$ obeys relations (3.5.13), (3.5.14), (3.5.15) with $\alpha l(t)$ playing the part of $l(t)$ by virtue of Lemma 3.5.2. The sequence $(g(n))$ is strongly uniformly distributed by virtue of Lemma 3.5.4. Let $m = [\alpha]-1$ and $g_m(t) = g(h_1, \dots, h_m, t)$ for fixed $h_1, \dots, h_m \in \mathbf{N}$. By virtue of Lemma 3.5.3, the hypotheses of lemma 3.5.4 are fulfilled, where the part of $g(t)$ is played by the function $g_m(t)$, and the part of $l(t)$, the function $h\alpha^{[m+2]} l(t)$. Therefore, the sequence $(g(h_1, \dots, h_m, n))$ is strongly uniformly distributed. From Lemma 3.5.1 it follows that the sequence $(g(n))$ is hence strongly uniformly distributed. The theorem is proved. □

Next, we consider the case where $\alpha \in \mathbf{N}$ in relation (3.5.5). Let I_m, $m \in \mathbf{N}$, stand for the set of all slowly varying at infinity functions $l(t)$ such that (3.5.5) holds for all $n = 1, \dots, m$. We set

$$I = \bigcap_{m \in \mathbf{N}} I_m.$$

THEOREM 3.5.3. *Let* (3.5.4) *hold for some* $\alpha \in \mathbf{N}$ *and a slowly varying at infinity function* $l(t)$. *If*

$$l'(t) = \delta t^{-1} l_1(t), \tag{3.5.23}$$

for any $t > 0$, *where* $\delta = 1$ *or* $\delta = -1$, $l_1(t) \in I_{\alpha+1}$, *and*

$$l_1(t) = o(l(t)), \qquad t \to \infty, \tag{3.5.24}$$

then the sequence $(g(n))$ *is strongly uniformly distributed.*

Let us formulate some corollaries to this theorem.

COROLLARY 3.5.2. *The sequence* $(n^\alpha \ln^\beta n)$ *is strongly uniformly distributed for any* $\alpha \in \mathbf{N}$ *and* $\beta \neq 0$.

COROLLARY 3.5.3. *The sequence* $(n^\alpha \ln^\gamma \ln n)$ *is strongly uniformly distributed for any* $\alpha \in \mathbf{N}$ *and* $\gamma \neq 0$.

COROLLARY 3.5.4. *The sequence* $(n^\alpha \ln^\beta n \ln^\gamma \ln n)$ *is strongly uniformly distributed for any* $\alpha \in \mathbf{N}$ *and* $\beta, \gamma \neq 0$.

COROLLARY 3.5.5. *The sequence* $(n^\alpha \exp(c \ln^\beta n))$ *is strongly uniformly distributed for any* $c \neq 0$, $\beta < 1$, *and* $\alpha \in \mathbf{N}$.

In what follows, for the sake of definiteness we let $\delta = 1$ (the case $\delta = -1$ is treated similarly). We prove Theorem 3.5.3 and its corollaries with the use of the following ten lemmas.

LEMMA 3.5.5. *Let* (3.5.4) *hold with* $\alpha \notin N \cup \{0\}$. *If* $l(t) \in I_m$ *for some* $m \in \mathbf{N}$, *then*

$$g^{(m)}(t) \sim \alpha^{[m]} t^{\alpha-m} l(t), \qquad t \to \infty. \tag{3.5.25}$$

LEMMA 3.5.6. *Under the hypotheses of Theorem* 3.5.3, $l(t) \in I_{\alpha+2}$.

LEMMA 3.5.7. *Under the hypotheses of Theorem* 3.5.3,

$$g^{(\alpha)}(t) \sim \alpha!\, l(t), \tag{3.5.26}$$

$$g^{(\alpha+1)}(t) \sim \alpha!\, t^{-1} l_1(t), \tag{3.5.27}$$

$$g^{(\alpha+2)}(t) \sim -\alpha!\, t^{-2} l_1(t), \tag{3.5.28}$$

as $t \to \infty$.

LEMMA 3.5.8. *Let* $m = \alpha - 1$, $h_1, \dots, h_m \in \mathbf{N}$, $g_m(t) = g(h_1, \dots, h_m, t)$, *and let the hypotheses of Theorem* 3.5.3 *be fulfilled. Then*

$$g'_m(t) \sim h\alpha!\, l(t),$$
$$g''_m(t) \sim h\alpha!\, t^{-1} l_1(t),$$
$$g'''_m(t) \sim -h\alpha!\, t^{-2} l_1(t),$$

as $t \to \infty$, *where* $h = h_1 \cdots h_m$.

LEMMA 3.5.9. *Let* $l(t)$ *be slowly varying at infinity. Then there exist functions* $r(t)$ *and* $s(t)$ *such that* $r(t) \to \infty$, $s(t) \to 0$ *as* $t \to \infty$, *and*

$$l(\lambda t)/l(t) \to 1, \qquad t \to \infty, \tag{3.5.29}$$

uniformly in $\lambda \in [s(t), r(t)]$.

LEMMA 3.5.10. *Let* (3.5.26), (3.5.27), *and* (3.5.28) *hold for some function* $g(t)$, $\alpha = 1$, *and let the functions* $l(t)$, $l_1(t)$ *be slowly varying at infinity. Then the sequence* $(g(n))$ *is strongly uniformly distributed.*

LEMMA 3.5.11. *A function* $l(t) \in I_m$ *for some* $m \in \mathbf{N}$ *if and only if*

$$\varphi^{(n)}(t) = o(1), \qquad t \to \infty, \tag{3.5.30}$$

for all $n = 1, \dots, m$, *where*

$$\varphi(t) = \ln l(e^t).$$

LEMMA 3.5.12. *Let* $l(t) \in I_m$, *where* $m \in \mathbf{N}$. *Then* $l^\beta(t) \in I_m$ *for any* $\beta \in \mathbf{R}$.

LEMMA 3.5.13. *Let* $l_1(t), l_2(t) \in I_m$, *where* $m \in \mathbf{N}$. *Then* $l(t) = l_1(t) l_2(t) \in I_m$.

LEMMA 3.5.14. *The functions* $\ln^\beta t$, $\ln^\gamma \ln t$ *belong to* I *for any real* β *and* γ.

The proof of Lemma 3.5.5 repeats that of Lemma 3.5.2.

PROOF OF LEMMA 3.5.6. From (3.5.23) and (3.5.24) it follows that

$$l'(t) = t^{-1} l_1(t) = o(t^{-1} l(t)), \qquad t \to \infty.$$

By virtue of Lemma 3.5.5, for $m = 2, \dots, \alpha + 2$, $n = m - 1$ we find that

$$\frac{d^m}{dt^m} l(t) = \frac{d^n}{dt^n} (l_1(t)/t) \sim (-1)^n n!\, t^{-n-1} l_1(t) = o(t^{-m} l(t)) \tag{3.5.31}$$

as $t \to \infty$. Therefore, $l(t) \in I_{\alpha+2}$. The lemma is true. □

PROOF OF LEMMA 3.5.7. The m-fold differentiation of equality (3.5.4), $m \le \alpha$, yields

$$g^{(m)}(t) = \sum_{k=0}^{m} \binom{\alpha}{k} \alpha^{[m-k]} t^{\alpha-(m-k)} l^{(k)}(t).$$

For $m = \alpha$ we obtain

$$g^{(\alpha)}(t) = \sum_{k=0}^{m} \binom{\alpha}{k} \alpha^{[\alpha-k]} t^k l^{(k)}(t). \tag{3.5.32}$$

From Lemma 3.5.6 it follows that for $k = 1, \dots, \alpha$

$$l^{(k)}(t) = o(t^{-k} l(t)), \qquad t \to \infty.$$

So, from (3.5.32) we arrive at

$$g^{(\alpha)}(t) \sim \alpha^{[\alpha]} \binom{\alpha}{0} l^{(0)}(t) = \alpha!\, l(t), \qquad t \to \infty,$$

and (3.5.26) is proved. The differentiation of (3.5.32) yields

$$g^{(\alpha+1)}(t) = \sum_{k=0}^{\alpha} \binom{\alpha+1}{k+1} \alpha^{[\alpha-k]} t^k l^{(k+1)}(t). \tag{3.5.33}$$

From (3.5.31) and (3.5.33) it follows that

$$\begin{aligned} g^{(\alpha+1)}(t) &\sim \frac{l_1(t)}{t} \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha+1}{k+1} \alpha^{[\alpha-k]} k! \\ &= \frac{l_1(t)}{t} \alpha! \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha+1}{k+1} = l_1(t) t^{-1} \alpha!\, \lambda, \end{aligned} \tag{3.5.34}$$

where

$$\lambda = -\sum_{k=0}^{\alpha} (-1)^{k+1} \binom{\alpha+1}{k+1}.$$

Let us calculate λ. We see that

$$\lambda = -\sum_{i=1}^{\alpha+1} (-1)^i \binom{\alpha+1}{i} = -\sum_{i=1}^{\beta} (-1)^i \binom{\beta}{i},$$

where $\beta = \alpha + 1$. For any real s,

$$\sum_{i=0}^{\beta} \binom{\beta}{i} s^i = (1+s)^\beta.$$

Therefore,

$$\sum_{i=0}^{\beta} \binom{\beta}{i} (-1)^i = (1-1)^\beta = 0.$$

Hence we find that $1 - \lambda = 0$, that is, $\lambda = 1$. Therefore, (3.5.27) follows from (3.5.34). The differentiation of (3.5.34) yields

$$g^{(\alpha+2)}(t) = \sum_{k=0}^{\alpha} \binom{\alpha+2}{k+2} \alpha^{[\alpha-k]} t^k l^{(k+2)}(t).$$

Taking (3.5.31) into account, hence we obtain

$$\begin{aligned} g^{(\alpha+2)}(t) &\sim \sum_{k=0}^{\alpha} \binom{\alpha+2}{k+2} \alpha^{[\alpha-k]} t^{-k} t^{k+2} (-1)^{k+1} (k+1)!\, l_1(t) \\ &= \alpha! \frac{l_1(t)}{t^2} \sum_{k=0}^{\alpha} \binom{\alpha+2}{k+2} (-1)^{k+1} (k+1) = \alpha!\, t^{-2} l_1(t) \mu, \end{aligned} \tag{3.5.35}$$

where

$$\mu = \sum_{k=0}^{\alpha} \binom{\alpha+2}{k+2} (-1)^{k+1} (k+1) = -\sum_{j=2}^{\gamma} \binom{\gamma}{j} (-1)^j (j-1), \qquad \gamma = \alpha + 2.$$

We observe that

$$\sum_{j=0}^{\gamma} \binom{\gamma}{j} s^{j+1} = s(s+1)^{\gamma}.$$

Differentiation of this equality yields

$$\sum_{j=0}^{\gamma} \binom{\gamma}{j} (j+1) s^j = (s+1)^{\gamma-1} (1 + (1+\gamma)s),$$

hence we obtain

$$\begin{aligned} \sum_{j=0}^{\gamma} \binom{\gamma}{j} (j-1) s^j &= (s+1)^{\gamma-1} (1 + (1+\gamma)s) - 2 \sum_{j=0}^{\gamma} \binom{\gamma}{j} s^j \\ &= (s+1)^{\gamma-1} (1 + (1+\gamma)s) - 2(1+s)^{\gamma}. \end{aligned}$$

Therefore,

$$-\sum_{j=0}^{\gamma} (j-1)(-1)^j = 0$$

and

$$\mu = -\sum_{j=2}^{\gamma} (j-1)(-1)^j = -1 + 0 = -1. \tag{3.5.36}$$

Relations (3.5.35) and (3.5.36) imply (3.5.28). The lemma is thus proved. □

The proof of Lemma 3.5.8 repeats that of Lemma 3.5.3.

PROOF OF LEMMA 3.5.9. By the theorem on integral representation of a slowly varying function (Seneta, 1976, Section 1.2), there exist $b > 0$ and functions $\eta(t)$, $\varepsilon(t)$ defined for $t \geq b$ such that $\eta(t) \to c$, $\varepsilon(t) \to 0$ with some constant c as $t \to \infty$, and

$$l(t) = \exp\left(\eta(t) + \int_b^t \frac{\varepsilon(u)}{u}\, du \right), \qquad \forall t \geq b.$$

Without loss of generality we assume that $\lambda \geq 1$. Then

$$\frac{l(\lambda t)}{l(t)} = \exp\left(\eta(\lambda t) - \eta(t) + \int_t^{\lambda t} \frac{\varepsilon(u)}{u}\, du\right), \qquad t \geq b. \tag{3.5.37}$$

We set

$$r(t) = \exp(1/\sqrt{\delta(t)}),$$

where

$$\delta(t) = \sup_{u \geq t} |\varepsilon(u)|.$$

Then

$$\left|\int_t^{\lambda t} \frac{\varepsilon(u)}{u}\, du\right| \leq \delta(t) \ln \lambda \leq \sqrt{\delta(t)} \to 0, \qquad t \to \infty,$$

uniformly in $\lambda \in [1, r(t)]$. Relation (3.5.29) follows from (3.5.37) and the last inequality. The lemma is thus proved. □

PROOF OF LEMMA 3.5.10. In accordance with Theorem 3.4.1 we check (3.4.2) for the sequence $x_n = g(n)$. We assume that $a \neq 0$. As before, we set $\theta = |b/a|$, $f(t) = ag(t) + bg(m-t)$. We first consider the case where $b/a \geq 0$. In accordance with Lemma 3.5.9, we choose a sequence $s(n) \to 0$, $n \to \infty$, in such a way that (3.5.29) holds and $ns(n) \in \mathbf{N}$. Making use of (3.5.26) with $\alpha = 1$, with the use of Lemma 3.5.9 for $t \in [ns(n), n - ns(n)]$ we obtain

$$\begin{aligned} f'(t) &= a(g'(t) - \theta g'(m-t)) \\ &= O(l(t)) + O(l(m-t)) = O(l(n)), \qquad n \to \infty, \end{aligned} \tag{3.5.38}$$

for an arbitrary sequence $m = m(n)$ such that $m \geq n$, $m = O(n)$ as $n \to \infty$. As follows from (3.5.27) and (3.5.28), there exists a constant M such that

$$|f''(t)| = |a|(g''(t) + \theta g''(m-t)) \geq |a| g''(t) \geq g''(n) \asymp n^{-1} l_1(n), \tag{3.5.39}$$

as $n \to \infty$ for $m \geq n \geq M$ and $t \in [ns(n), n - ns(n)]$. From (3.5.38), (3.5.39), and Theorem 3.4.3 it follows that

$$\begin{aligned} \left|\sum_{t=ns(n)}^{n-s(n)} \exp(2\pi i f(t))\right| &\leq (|f'(ns(n))| + |f'(n - ns(n))| + 2)\left(\frac{4}{\sqrt{\rho}} + 3\right) \\ &= O(l(n)) O\left(\frac{1}{\sqrt{n^{-1} l_1(n)}}\right) \\ &= O\left(\frac{\sqrt{n}\, l(n)}{\sqrt{l_1(n)}}\right) = o(n), \qquad n \to \infty, \end{aligned}$$

for an arbitrary sequence $m = m(n)$ such that $m \geq n$, $m = O(n)$ as $n \to \infty$, where

$$\rho = \inf_{ns(n) \leq t \leq n - ns(n)} |f''(t)|.$$

Therefore,

$$\sum_{t=1}^{n} \exp(2\pi i f(t)) = o(n) + O(ns(n)) = o(n), \qquad n \to \infty,$$

for an arbitrary sequence $m = m(n)$ such that $m \geq n$, $m = O(n)$ as $n \to \infty$. Hypothesis (3.4.2) of Theorem 3.4.1 is thus validated in this case. For $b/a < 0$ the proof repeats literally that of Lemma 3.5.4. If $a = 0$, then relations (3.5.38), (3.5.39) are replaced by

$$f'(t) = -bg'(m-t) = O(l(m-t)) = O(l(n)), \qquad n \to \infty,$$
$$|f''(t)| = |b|g''(m-t) \geq |b|g''(m) \asymp n^{-1} l_1(n), \qquad n \to \infty,$$

respectively, which, as above, imply (3.4.2) for the sequence $x_n = g(n)$. The proof of the lemma is thus complete. □

PROOF OF LEMMA 3.5.11. Let $l(t) \in I_m$ and $\psi(t) = l(\exp(t))$. By formula (0.430) in (Gradshteyn, Ryzhik, 1980) for the nth derivative of a composite function, for $n = 1, \dots, n$ we obtain

$$\frac{d^n}{dt^n}\psi(t) = \sum_{i_1,\dots,i_n} l^{(k)}(\exp(t)) C(i_1, \dots, i_n) \prod_{j=1}^{n} (\exp(t))^{i_j},$$

where the summation is over all ordered tuples of non-negative integers $i_1, \dots, i_n$ such that

$$\sum_{j=1}^{n} j i_j = n, \qquad k = \sum_{j=1}^{n} i_j,$$

and $C(i_1, \dots, i_n)$ are some constants. Hence it follows that

$$\psi^{(n)}(t) = \sum_{i_1,\dots,i_n} o(e^{-tk} l(e^t)) e^{tk} = o(l(e^t)) = o(\psi(t)) \qquad (3.5.40)$$

as $t \to \infty$. Applying formula (0.430) in (Gradshteyn, Ryzhik, 1980) again to the function $\varphi(t) = \ln \psi(t)$, for $n = 1, \dots, m$ we find that

$$\varphi^{(n)}(t) = \sum_{i_1,\dots,i_n} \ln^{(k)}(\psi(t)) C(i_1, \dots, i_n) \prod_{j=1}^{n} (\psi^{(j)}(t))^{i_j}.$$

Making use of (3.5.40), hence we obtain

$$\varphi^{(n)}(t) = \sum_{i_1,\dots,i_n} (-1)^{k+1}(k-1)!(\psi(t))^{-k} C(i_1, \dots, i_n) o((\psi(t))^k) = o(1)$$

as $t \to \infty$. Relation (3.5.30) is thus proved. The proof of the reverse assertion repeats the proof of Lemma 1.1.1. □

PROOF OF LEMMA 3.5.12. Let

$$\varphi(t) = \ln l(\exp(t)), \qquad \Phi(t) = \ln(l^{\beta}(\exp(t))).$$

By virtue of Lemma 3.5.11, relation (3.5.30) holds true for any $n = 1, \dots, m$. Therefore, $\Phi^{(n)}(t) = o(1)$ as $t \to \infty$ for all $n = 1, \dots, m$. Consequently, $l^{\beta}(t) \in I_m$ by virtue of Lemma 3.5.11. □

PROOF OF LEMMA 3.5.13. Let $n = 1, \dots, m$. The n-fold differentiation of $l(t)$ yields

$$\frac{d^n}{dt^n} l(t) = \sum_{k=0}^{n} \binom{n}{k} l_1^{(k)}(t) l_2^{(n-k)}(t);$$

hence we obtain

$$\begin{aligned} l^{(n)}(t) &= \sum_{k=1}^{n-1} o(t^{-k} l_1(t)) o(t^{-(n-k)} l_2(t)) + o(l_1(t) t^{-n} l_2(t)) \\ &= o(t^{-n} l_1(t) l_2(t)), \qquad t \to \infty. \end{aligned}$$

Therefore, $l(t) \in I_m$. □

PROOF OF LEMMA 3.5.14. For any $n \in \mathbf{N}$ we observe that

$$\frac{d^n}{dt^n} \ln t = (-1)^{n-1}(n-1)!\, t^{-n} = o(t^{-n} \ln t), \qquad t \to \infty.$$

Therefore, $\ln t \in I$. By virtue of Lemma 3.5.12, $\ln^{\beta} t \in I$. For $\ln \ln t$ we see that

$$\frac{d}{dt} \ln \ln t = \frac{1}{t \ln t} = o(t^{-1} \ln \ln t), \qquad t \to \infty, \tag{3.5.41}$$

and for $m > 1$, $n = m - 1$ by formula (3.5.31) we obtain

$$\begin{aligned} \frac{d^m}{dt^m} \ln \ln t &= \frac{d^n}{dt^n}(t^{-1} \ln^{-1} t) \sim (-1)^n n!\, t^{-n-1} \ln^{-1} t \\ &= o(t^{-m} \ln \ln t), \qquad t \to \infty. \end{aligned} \tag{3.5.42}$$

From (3.5.41) and (3.5.42) it follows that $\ln \ln t \in I$. From Lemma 3.5.12 it follows that $\ln^{\gamma} \ln t \in I$. The lemma is thus proved. □

PROOF OF THEOREM 3.5.3. If $\alpha = 1$, then, by virtue of Lemma 3.5.10, Theorem 3.5.3 holds true. Let $\alpha > 1$ and

$$l(t) = h\alpha!\, l(t), \quad l_1(t) = h\alpha!\, l_1(t), \quad m = \alpha - 1, \quad g(t) = g_m(t).$$

From Lemma 3.5.8 and Lemma 3.5.10 it follows that the sequence $(g(h_1, \dots, h_m, n))$ is strongly uniformly distributed for any $h_1, \dots, h_m \in \mathbf{N}$. Lemma 3.5.1 now implies that the sequence $(g(n))$ is strongly uniformly distributed as well. Theorem 3.5.3 is proved. □

PROOF OF COROLLARY 3.5.2. For $l(t) = \ln^{\beta}(t)$ we see that

$$l'(t) = \beta \ln^{\beta-1}(t)/t.$$

By virtue of Lemma 3.5.14, the function $l_1(t) = |\beta| \ln^{\beta-1}(t) \in I$. Furthermore, it is clear that $l_1(t) = o(l(t))$ as $t \to \infty$. □

PROOF OF COROLLARY 3.5.3. Let $L(t) = \ln \ln t$. Then

$$l(t) = \ln^{\gamma} \ln t = L^{\gamma}(t).$$

We observe that

$$l'(t) = \gamma L^{\gamma-1}(t) L'(t) = \frac{\gamma L^{\gamma-1}(t)}{t \ln t}.$$

Therefore,

$$l_1(t) = |\gamma| L^{\gamma-1}(t) \ln^{-1}(t).$$

From Lemma 3.5.12 it follows that $\ln^{-1}(t) \in I$. Lemma 3.5.14 implies $L^{\gamma-1}(t) \in I$. So, by virtue of Lemma 3.5.13, $l_1(t) \in I$. □

PROOF OF COROLLARY 3.5.4. We consider the case where $\alpha > 0$, $\beta > 0$ (the remaining cases are treated similarly). We see that

$$l'(t) = \frac{\beta \ln^{\beta-1} t \ln^{\gamma} \ln t}{t} + \gamma \frac{ln^{\beta} t \ln^{\gamma-1} \ln t}{t \ln t}.$$

Therefore,

$$\begin{aligned} l_1(t) &= \beta \ln^{\beta-1} t \ln^{\gamma} \ln t + \gamma \ln^{\beta} t \ln^{\gamma-1} \ln t \\ &\sim \beta \ln^{\beta-1} t \ln^{\gamma} \ln t, \qquad t \to \infty. \end{aligned}$$

It is clear that $l_1(t)$ is slowly varying at infinity and $l_1(t) = o(l(t))$ as $t \to \infty$. We observe that $l_1(t) = l_2(t) + l_3(t)$, where

$$l_2(t) = \beta \ln^{\beta-1} t \ln^{\gamma} \ln t, \qquad l_3(t) = \gamma \ln^{\beta} t \ln^{\gamma-1} \ln t.$$

By virtue of Lemmas 3.5.13, 3.5.14, the functions $l_2(t), l_3(t) \in I$. Therefore,

$$l_1^{(n)}(t) = l_2^{(n)}(t) + l_3^{(n)}(t) = o(t^{-n} l_2(t)) + o(t^{-n} l_3(t)) = o(t^{-n} l_1(t))$$

for any $n \in \mathbf{N}$ as $t \to \infty$. Thus, $l_1(t) \in I$. □

PROOF OF COROLLARY 3.5.5. We observe that

$$\begin{aligned} l'(t) &= c\beta l(t) \ln^{\beta-1}(t)/t, \\ l_1(t) &= |c\beta| l(t) \ln^{\beta-1}(t). \end{aligned}$$

For $\beta < 1$,

$$\frac{t l'(t)}{l(t)} = c\beta \ln^{\beta-1}(t) \to 0, \qquad t \to \infty.$$

Hence $l(t)$ is slowly varying at infinity. Let us check whether $l(t) \in I$ or not. It is easy to see that

$$\varphi(t) = \ln l(\exp(t)) = ct^{\beta},$$

so that

$$\varphi^{(n)}(t) = c\beta^{[n]}t^{\beta-n} = o(1), \qquad t \to \infty,$$

for any $n \in \mathbf{N}$. Therefore, $l(t) \in I$ by virtue of Lemma 3.5.11. The function $l_1(t) \in I$ as the product of two functions of I (Lemma 3.5.13). □

3.6. Random sets A

In this section we study the case where the set A is random itself. Namely, let a sequence $(\xi_n,\ n \in \mathbf{N})$ of independent Bernoulli random variables be given. By this sequence, we construct the random set A as follows: an element $n \in \mathbf{N}$ is contained in the set A if and only if $\xi_n = 1$. For each realisation of the set A, as in the beginning of Section 3.1, we introduce the set $T_n = T_n(A)$ and the random variables ζ_{nm} and ζ_n, $n, m \in \mathbf{N}$. We recall that T_n is the set of permutations of degree n whose cycle lengths belong to the set A, ζ_{nm} is the number of cycles of a random permutation uniformly distributed on T_n of length $m \in A$, and ζ_n is the total number of its cycles (if T_n appears to be empty, then ζ_{nm} and ζ_n are set to zero).

We set

$$p_n = \mathsf{P}\{\xi_n = 1\}, \quad n \in \mathbf{N}, \quad p_0 = 0.$$

We assume that, as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} p_i \to \sigma > 0, \tag{3.6.1}$$

and that for any fixed $c > 1$

$$\frac{1}{n}\sum_{i=1}^{n} p_i p_{m-i} \to \sigma^2 \tag{3.6.2}$$

uniformly in $m \in [n, cn]$. The following analogues of Theorems 3.3.1–3.3.3 are true.

THEOREM 3.6.1. *Let relations* (3.6.1) *and* (3.6.2) *be true. Then, as* $n \to \infty$,

$$\frac{|T_n(A)|}{n!\,L(n)} n^{1-\sigma} \to \frac{e^{\sigma\gamma}}{\Gamma(\sigma)}$$

almost surely (a.s.), where

$$L(n) = \exp(l(n) - \sigma \ln n), \qquad l(n) = \frac{1}{n}\sum_{k=1}^{n} \xi_k,$$

γ *is the Euler constant,* $\Gamma(\cdot)$ *is the gamma function.*

THEOREM 3.6.2. *Let relations* (3.6.1) *and* (3.6.2) *be true. Then for any fixed* $x \in \mathbf{R}$, *as* $n \to \infty$,

$$\mathsf{P}\{(\zeta_n - l(n))/\sqrt{\sigma \ln n} \le x \mid A\} \xrightarrow{\text{a.s.}} \Phi(x),$$

where $\Phi(x)$ *is the distribution function of the standard normal law.*

THEOREM 3.6.3. *Let relations* (3.6.1) *and* (3.6.2) *be true. Then for any fixed* $m \in \mathbf{N}$ *and* $k \in \mathbf{N}$, *as* $n \to \infty$

$$\mathsf{P}\{\zeta_{nm} = k \mid A\} \xrightarrow{\text{a.s.}} \frac{\xi_n \exp(-1/m)}{m^k k!}$$

and

$$\mathsf{P}\{\zeta_{nm} = 0 \mid A\} \xrightarrow{\text{a.s.}} 1 - \xi_m + \xi_m \exp(-1/m).$$

Theorems 3.6.1–3.6.3 validate the hypothesis formulated by V. F. Kolchin in 1989 at the seminar in the Steklov Institute of Mathematics. The hypothesis states that analogues of Theorems 3.3.1–3.3.3 are true in the case of identically distributed random variables ξ_n, $n \in \mathbf{N}$. Theorems 3.6.1–3.6.3 are published in (Yakymiv, 2000). Let us formulate two corollaries.

COROLLARY 3.6.1. *As* $n \to \infty$, *let*

$$p_n \to \sigma > 0. \tag{3.6.3}$$

Then relations (3.6.1) *and* (3.6.2) *are true.*

For $p_n = \sigma$ for all $n \in \mathbf{N}$ hence it follows that the above hypothesis is valid.

COROLLARY 3.6.2. *Let* B *be some fixed subset of the set* $\mathbf{N}$ *of asymptotic density zero, that is, as* $n \to \infty$

$$\frac{1}{n}|k : k \le n,\ k \in B| \to 0.$$

If

$$p_n \to \sigma > 0, \quad n \to \infty, \quad n \in \mathbf{N} \setminus B, \tag{3.6.4}$$

then relations (3.6.1) *and* (3.6.2) *hold.*

Corollary 3.6.1 demonstrates that Theorems 3.6.1–3.6.3 are true no matter how the numbers p_n behave on any finite fixed domain of variation of n; it suffices that relation (3.6.3) is true.

Let us turn to proving Theorems 3.6.1–3.6.3.

PROOF OF THEOREMS 3.6.1–3.6.3. By virtue of Theorems 3.3.1–3.3.3, in order to prove our theorems we have to show that, as $n \to \infty$,

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k \xrightarrow{\text{a.s.}} \sigma \tag{3.6.5}$$

and that for an arbitrary fixed $c > 1$

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k \xi_{m-k} \xrightarrow{\text{a.s.}} \sigma^2 \tag{3.6.6}$$

uniformly in $m \in [n, cn]$.

Relation (3.6.5) immediately follows from (3.6.1) and the strong law of large numbers for the sequence ξ_n, $n \in \mathbf{N}$. Relation (3.6.6) does not follow from (3.6.2) and the strong law of large numbers, so we prove it below.

In view of (3.6.2), relation (3.6.6) is equivalent to

$$\frac{1}{n}\sum_{k=1}^{n} X(k, m) \xrightarrow{\text{a.s.}} 0 \tag{3.6.7}$$

uniformly in $m \in [n, cn]$, where

$$X(k, m) = \xi_k \xi_{m-k} - p_k p_{m-k}.$$

Next, let us show that the inequality

$$\mathbf{E}\left(\sum_{k=1}^{n} X(k, m)\right)^6 \le c_1 n^3 \tag{3.6.8}$$

where $c_1 = 111 \cdot 6!$ holds true for all $n \in \mathbf{N}$ and $m \ge n$. First, treating (3.6.8) as if it were true, we derive relation (3.6.7) from it. Let some $\varepsilon \in (0, 1]$ be fixed. We set

$$A_{m,n}(\varepsilon) = \left\{\omega\colon \left|\frac{1}{n}\sum_{k=1}^{n} X(k, m)\right| > \varepsilon\right\},$$

$$B(\varepsilon) = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{m=n}^{[cn]} A_{mn}(\varepsilon).$$

For any $k \in \mathbf{N}$, by (3.6.8) and the Chebyshev inequality we obtain

$$\mathbf{P}(B(\varepsilon)) \le \sum_{n=k}^{\infty}\sum_{m=n}^{\infty} \mathbf{P}(A_{mn}(\varepsilon)) \le \sum_{n=k}^{\infty}\sum_{m=n}^{\infty} \frac{cn^3}{(\varepsilon n)^6},$$

and the right-hand side of the inequality tends to zero as $k \to \infty$. Hence it follows that

$$\mathbf{P}(B(\varepsilon)) = 0, \tag{3.6.9}$$

that is, only finitely many events $A_{mn}(\varepsilon)$ can occur with probability one. We set

$$B = \bigcup_{\varepsilon \le 1} B(\varepsilon).$$

Let B' stand for the set of those elementary outcomes ω for which $\frac{1}{n}\sum_{i=1}^{n} X(i, m)$ does not tend to zero as $n \to \infty$ uniformly in $m \in [n, cn]$. Let us demonstrate that $B = B'$.

Let $\omega \in B$. Then there exists $\varepsilon \in (0, 1]$ such that $\omega \in B(\varepsilon)$, in other words, for an arbitrary $k \in \mathbf{N}$ there exist $n \geq k$ and $m \in [n, cn]$ such that

$$\left| \frac{1}{n} \sum_{i=1}^{n} X(i, m) \right| > \varepsilon,$$

so that $\frac{1}{n} \sum_{i=1}^{n} X(i, m)$ does not tend to zero as $n \to \infty$ uniformly in $m \in [n, cn]$.

Now let $\omega \in B'$. Then for any $k \in \mathbf{N}$ and any $\varepsilon \in (0, 1]$ there exist $n \geq k$ and $m \in [n, cn]$ such that

$$\left| \frac{1}{n} \sum_{i=1}^{n} X(i, m) \right| > \varepsilon,$$

that is, $\omega \in B$. Thus, $B = B'$.

For $0 < \varepsilon_1 < \varepsilon_2 \leq 1$, the inclusion $B(\varepsilon_1) \subseteq B(\varepsilon_2)$ is true, therefore,

$$B = \bigcup_{k=1}^{\infty} B(1/k),$$

so $\mathsf{P}(B') = \mathsf{P}(B) = 0$ by (3.6.9), which yields (3.6.7).

In order to prove Theorems 3.6.1–3.6.3 it remains to validate inequality (3.6.8). To do this, we first note that

$$\mathsf{E} \left(\sum_{k=1}^{n} X_k \right)^6 \leq 6!\, (\Sigma_1 + \cdots + \Sigma_{11}) \tag{3.6.10}$$

(for the sake of brevity we write X_k instead of $X(k, m)$), where

$$\Sigma_1 = \sum_{i=1}^{n} \mathsf{E} X_i^6,$$

$$\Sigma_2 = \sum |\mathsf{E} X_i^5 X_j|$$

(in the second sum the summation is over all i, j from 1 to n such that $i \neq j$),

$$\Sigma_3 = \sum |\mathsf{E} X_i^4 X_j X_k|$$

(the summation is over all i, j, k from 1 to n such that $i \neq j, i \neq k, j < k$),

$$\Sigma_4 = \sum |\mathsf{E} X_i^4 X_j^2|$$

(as for Σ_2, the summation is over all i, j from 1 to n such that $i \neq j$),

$$\Sigma_5 = \sum |\mathsf{E} X_i^3 X_j^2 X_k|$$

(the summation is over all i, j, k from 1 to n such that $i \neq j, i \neq k, j \neq k$),

$$\Sigma_6 = \sum |\mathbf{E} X_i^3 X_j^3|$$

(the summation is over all i, j from 1 to n such that $i \neq j$),

$$\Sigma_7 = \sum |\mathbf{E} X_i^2 X_j^2 X_k^2|$$

(the summation is over all i, j, k from 1 to n such that $i \neq j, i \neq k, j \neq k$),

$$\Sigma_8 = \sum |\mathbf{E} X_i^3 X_j X_k X_l|$$

(the summation is over all i, j, k, l from 1 to n such that $i \neq j, i \neq k, i \neq l, j < k < l$),

$$\Sigma_9 = \sum |\mathbf{E} X_i^2 X_j^2 X_k X_l|$$

(the summation is over all i, j, k, l from 1 to n such that $i \neq j, i \neq k, i \neq l, j \neq k, j \neq l, k < l$),

$$\Sigma_{10} = \sum |\mathbf{E} X_i^2 X_j X_k X_l X_u|$$

(the summation is over all i, j, k, l, u from 1 to n such that $i \neq j, i \neq k, i \neq l, i \neq u, j < k < l < u$),

$$\Sigma_{11} = \sum |\mathbf{E} X_i X_j X_k X_l X_u X_v|$$

(the summation is over all i, j, k, l, u, v from 1 to n such that $i < j < k < l < u$).

By virtue of the trivial inequality

$$|X_i| \leq 1, \qquad i = 1, \ldots, n, \tag{3.6.11}$$

we easily see that

$$\Sigma_k \leq n^3, \qquad k = 1, \ldots, 7. \tag{3.6.12}$$

Let us find a bound for Σ_8. We set

$$\Sigma_{8,1} = \sum |\mathbf{E} X_i^3 X_j X_k X_l|,$$

where the summation is over the same i, j, k, l as in Σ_8 but under the additional constraints $l \neq m-i, l \neq m-j, l \neq m-l$. It is not difficult to see that $\Sigma_{8,1} = 0$, because under these constraints the random variable X_l is independent of $X_i^3 X_j X_k$ and $\mathbf{E} X_l = 0$. Therefore,

$$\Sigma_8 \leq \Sigma_{8,2} + \Sigma_{8,3} + \Sigma_{8,4} + \Sigma_{8,5},$$

where the summation in the sums $\Sigma_{8,2}, \dots, \Sigma_{8,5}$ is under the same constraints as in Σ_8, but under the additional one $l = m - i$, $l = m - j$, $l = m - k$, $l = m - l$, respectively. It is not difficult to see that

$$\Sigma_{8,2} \leq \Sigma_3 \leq n^3,$$
$$\Sigma_{8,3} \leq \Sigma_5 \leq n^3,$$
$$\Sigma_{8,4} \leq \Sigma_5 \leq n^3.$$

Finally, by virtue of (3.6.11)

$$\Sigma_{8,5} \leq \sum_{i,j,k=1}^{n} |\mathbf{E} X_i^3 X_j X_k X_{m/2}| \leq n^3,$$

(for non-integer a we set $X_a = 0$). We thus arrive at the inequality

$$\Sigma_8 \leq 4n^3. \tag{3.6.13}$$

Next, let us find a bound for Σ_9. To do this, we first observe that the sum

$$\Sigma_{9,1} = \sum |\mathbf{E} X_i^2 X_j^2 X_k X_l|,$$

where the summation is over the same i, j, k, l as in the sum Σ_9 but under the additional constraint that $l \neq m - i$, $l \neq m - j$, $l \neq m - k$, $l \neq m - l$, vanishes because under these constraints X_l is independent of $X_i^2 X_j^2 X_k$ and $\mathbf{E} X_l = 0$. Therefore,

$$\Sigma_9 \leq \Sigma_{9,2} + \Sigma_{9,3} + \Sigma_{9,4} + \Sigma_{9,5},$$

where the summation in the sums $\Sigma_{9,2}, \dots, \Sigma_{9,5}$ is over the same i, j, k, l as in Σ_9 but under the additional constraint that $l = m - i$, $l = m - j$, $l = m - k$, $l = m - l$, respectively. It is not difficult to see that

$$\Sigma_{9,2} \leq \Sigma_5 \leq n^3,$$
$$\Sigma_{9,3} \leq \Sigma_5 \leq n^3,$$
$$\Sigma_{9,4} \leq \Sigma_7 \leq n^3,$$
$$\Sigma_{9,5} \leq \sum_{i,j,k=1}^{n} |\mathbf{E} X_i^2 X_j^2 X_k X_{m/2}| \leq n^3.$$

Combining these bounds we find that

$$\Sigma_9 \leq 4n^3. \tag{3.6.14}$$

As above, the sum

$$\Sigma_{10,1} = \sum |\mathbf{E} X_i^2 X_j X_k X_l X_u|,$$

where the summation is over the same i, j, k, l, u as in the sum Σ_{10} but under the additional constraints that $u \neq m - i$, $u \neq m - j$, $u \neq m - k$, $u \neq m - l$, $u \neq m - u$, vanishes. Therefore,

$$\Sigma_{10} \leq \Sigma_{10,2} + \Sigma_{10,3} + \Sigma_{10,4} + \Sigma_{10,5} + \Sigma_{10,6},$$

where the summation in the sums $\Sigma_{10,2}, \ldots \Sigma_{10,6}$ is over the same i, j, k, l, u as in Σ_{10} but under the additional constraint that $u = m - i$, $u = m - j$, $u = m - k$, $u = m - l$, $u = m - u$ respectively. It is clear that

$$\Sigma_{10,2} \leq \Sigma_8 \leq 4n^3,$$

and for $t = 3, 4, 5$

$$\Sigma_{10,t} \leq \Sigma_9 \leq 4n^3.$$

We observe that

$$\Sigma_{10,6} = \sum |\mathbf{E} X_i^2 X_j X_k X_l X_{m/2}| = 0,$$

because $j < k < l < m/2, i \neq m/2$ in the summation domain, so

$$\mathbf{E} X_i^2 X_j X_k X_l X_{m/2} = \mathbf{E} X_i^2 X_j X_k X_l \mathbf{E} X_{m/2}$$

and

$$\mathbf{E} X_j = \mathbf{E} X_k = \mathbf{E} X_l = 0.$$

Out of three subscripts j, k, l, only one can be equal to i, and one, to $m - i$, whereas the random variable labelled with the third of these subscripts is independent of the others, and $\mathbf{E} X_i^2 X_j X_k X_l$ is equal to zero. Combining these bounds, we arrive at

$$\Sigma_{10} \leq 4n^3 + 3 \cdot 4n^3 = 16n^3. \tag{3.6.15}$$

It remains to find a bound for Σ_{11}. First we observe that the sum

$$\Sigma_{11,1} = \sum |\mathbf{E} X_i X_j X_k X_l X_u X_v|,$$

where the summation is over all i, j, k, l, u, v from 1 to n such that $i < j < k < l < u < v$ and $v \neq m - i$, $v \neq m - j$, $v \neq m - k$, $v \neq m - l$, $v \neq m - u$, $v \neq m - v$, vanishes. Therefore,

$$\Sigma_{11} \leq \sum_{t=2}^{7} \Sigma_{11,t},$$

where the summation in the sums $\Sigma_{11,2}, \ldots, \Sigma_{11,7}$ is over the same i, j, k, l, u, v as in the sum Σ_{11} but under the additional constraint that $v = m - i$, $v = m - j$, $v = m - k$, $v = m - l$, $v = m - u$, $v = m - v$ respectively. It is clear that

$$\Sigma_{11,t} \leq \Sigma_{10} \leq 16n^3, \qquad t = 2, \ldots, 6.$$

We also see that

$$\Sigma_{11,7} = \sum |\mathbf{E} X_i X_j X_k X_l X_u X_{m/2}| = 0$$

(the summation is over all i, j, k, l, u from 1 to n such that $i < j < k < l < u < m/2$), because

$$\mathbf{E} X_i = \mathbf{E} X_j = \mathbf{E} X_k = \mathbf{E} X_l = \mathbf{E} X_u = 0$$

and the random variables X_i, X_j, X_k, X_l, X_u, $X_{m/2}$ are independent, since the sum of any pair of subscripts of $\{i, j, k, l, u\}$ is less than m. Thus, we arrive at the bound

$$\Sigma_{11} \leq 80n^3. \tag{3.6.16}$$

Combining bounds (3.6.10) and (3.6.12)–(3.6.16), we obtain

$$\mathbf{E}\left(\sum_{k=1}^{n} X_k\right)^6 \leq 6!\, n^3(7 + 4 + 4 + 16 + 80) = 6!\, 111 n^3.$$

Theorems 3.6.1–3.6.3 are thus proved. □

4

Infinitely divisible distributions

4.1. Probabilities of large deviations

In this and the next section, we assume that the random variable ξ has an *infinitely divisible* distribution:

$$\mathbf{E}e^{it\xi} = \exp\left(i\gamma t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)G(dx)\right),$$

where $G(dx)$ is its Lévy spectral measure (Petrov, 1975, section 2.2, formula (2.12)) on $(-\infty, 0) \cup (0, \infty)$ (maybe unbounded in the neighbourhood of zero), and the integrals

$$\int_{-1}^{1} x^2 G(dx), \quad \int_{-\infty}^{-1} G(dx), \quad \int_{1}^{\infty} G(dx)$$

are finite, $\sigma \geq 0$, γ is some real number, and $t \in (-\infty, \infty)$. Here we study the asymptotic behaviour of the probability $\mathbf{P}\{\xi > t\}$ as $t \to \infty$ in terms of the spectral functions

$$q(t) = \int_{t}^{\infty} G(dx), \quad p(t) = \int_{-\infty}^{-t} G(dx), \quad t > 0.$$

Three theorems below and their corollaries are the main results of this section.

THEOREM 4.1.1. *Let*

$$\limsup_{t\to\infty} \frac{q(t)}{q(2t)} < \infty, \tag{4.1.1}$$

that is, $q(t)$ is dominatedly varying at infinity (see Remark 1.6.1). Then, as $t \to \infty$,

$$\mathbf{P}\{\xi > t\} \overset{w}{\sim} q(t).$$

The definition of weak convergence of functions is given in the beginning of Section 1.6.

COROLLARY 4.1.1. *Let $q(t)$ be weakly oscillating at infinity, that is, $q(\tau)/q(t) \to 1$ as $t \to \infty$, $\tau/t \to 1$. Then*

$$\mathbf{P}\{\xi > t\} = (1 + o(1))q(t), \qquad t \to \infty.$$

THEOREM 4.1.2. *Let* $\mathbf{E}|\xi| < \infty$, (4.1.1) *hold, and for some* $a > 0$ *let the function* $q(t)$ *possess a continuous derivative* $q'(t)$ *which is concave on* $[a, \infty)$. *Then, as* $t \to \infty$,

$$\mathbf{P}\{\xi > t\} = q(t) - \mathbf{E}\xi q'(t) + o(q(t)/t).$$

COROLLARY 4.1.2. *Under the hypotheses of Theorem* 4.1.2,

$$\mathbf{P}\{\xi > t\} = q(t) + O(q(t)/t), \qquad t \to \infty.$$

COROLLARY 4.1.3. *Let* $\mathbf{E}|\xi| < \infty$, $q(t)$ *be regularly varying at infinity with index* $-\alpha \in (-\infty, -1)$, *and* $q''(t)$ *do not increase for* $t \geq a$. *Then, as* $t \to \infty$,

$$P\{\xi > t\} = q(t) + \alpha \mathbf{E}\xi q(t)/t + o(q(t)/t).$$

THEOREM 4.1.3. *Let* $q(t)$ *be regularly varying at infinity with index* $-\alpha \in (-1, 0]$, *and for some* $a > 0$ *let the function* $q(t)$ *possess a continuous derivative* $q'(t)$ *which is concave on* $[a, \infty)$. *Then, as* $t \to \infty$,

$$\mathbf{P}\{\xi > t\} = q(t) - \frac{\Gamma^2(1-\alpha)}{2\Gamma(2-2\alpha)} q^2(t) + O\left(\frac{q(t)}{t} \int_0^t p(u)\, du\right) + o(q^2(t)),$$

where $\Gamma(\cdot)$ *is the Euler gamma function.*

COROLLARY 4.1.4. *Let the hypotheses of Theorem* 4.1.3 *be satisfied and* $p(t)$ *be Lebesgue-integrable on* $[1, \infty)$. *Then, as* $t \to \infty$,

$$\mathbf{P}\{\xi > t\} = q(t) - \frac{\Gamma^2(1-\alpha)}{2\Gamma(2-2\alpha)} q^2(t) + o(q^2(t)).$$

For $\alpha = 1/2$, the second term in the last two asymptotic formulas vanishes.

A great body of studies are devoted to the asymptotic behaviour of infinitely divisible distributions at infinity: (Antonov, 1981; Kruglov, Antonov, 1982; Kruglov, Antonov, 1984; Baltrunas, Yakymiv, 2003; Zolotarev, 1961; Zolotarev, 1965; Kruglov, 1971; Kruglov, 1972; Kruglov, 1973; Kruglov, 1974a; Kruglov, 1974b; Ulanovskii, 1981; Sgibnev, 1991; Sgibnev, 1990; Yakymiv, 1987b; Yakymiv, 1990b; Yakymiv, 2002; Acosta, 1980; Acosta, Gine, 1979; Araujo, Gine, 1980; Embrechts *et al.*, 1979; Embrechts, Goldie, 1982; Grübel, 1983; Grübel, 1983; Grübel, 1987; Hikaru, 1989; Sato, 1999; Ramachandran, 1969; Ruegg, 1971; Yakymiv, 1997), just to name a few. Let us highlight the most typical results. In 1961, Zolotarev showed that if $\xi \geq 0$ and $q(t)$ regularly varied at infinity, then $\mathbf{P}\{\xi > t\} \sim q(t)$ as $t \to \infty$. In (Embrechts *et al.*, 1979), a necessary and sufficient condition for $\mathbf{P}\{\xi > t\} \sim q(t)$ as $t \to \infty$ was found. It consists of *subexponentiality* of the distribution of ξ. The subexponential distributions were introduced in (Chistyakov, 1965) and found application in the theory of branching processes, queue theory, renewal theory, and the theory of infinitely divisible distributions. We recall that the distribution of a random variable ξ is said to be *subexponential* if

$$\frac{1 - F * F(t)}{1 - F(t)} \to 2, \qquad t \to \infty,$$

where $F(t)$ is the distribution function of the random variable ξ (for more details, see (Bingham *et al.*, 1987)). In (Sgibnev, 1990) the case is considered where the last ratio tends to some constant, so one finds oneself in the context of the so-called distributions of *exponential type* introduced in (Chover *et al.*, 1973a; Chover *et al.*, 1973b)). By the way, (4.1.1) does not imply that the distribution of the random variable ξ is subexponential.

Let a positive decreasing function $\tau(t)$ obey the relation

$$-\int_0^t \tau(t-s)\,d\tau(s) = O(\tau(t)), \qquad t \to \infty.$$

In (Grübel, 1983) it is shown that

$$\mathbf{P}\{\xi > t\} = O(\tau(t)) \iff q(t) = O(\tau(t)), \qquad t \to \infty,$$

and the symbol O can be replaced by o.

Various bounds for the difference between $\mathbf{P}\{\xi > t\}$ and $q(t)$ are found in (Grübel, 1983; Grübel, 1987; Baltrunas, Omey, 1998; Baltrunas, Yakymiv, 2003; Yakymiv, 2001). In (Kruglov, 1984, Theorem 6.3) it is shown that if P is an infinitely divisible distribution in a Hilbert space with Lévy measure G, then the relations

$$\inf\{t\colon t > 0,\ G(\{x\colon \|x\| > t\}) = 0\} = \gamma$$

and

$$-\lim_{t\to\infty} \frac{\ln P(\{x\colon \|x\| > t\})}{t\ln(t+1)} = \frac{1}{\gamma}$$

are equivalent.

The case where the Lévy measure of an infinitely divisible distribution is concentrated on a finite set is quite thoroughly studied. The most general results are in (Kruglov, Antonov, 1982; Kruglov, Antonov, 1984). Some questions related to asymptotic behaviour of infinitely divisible distributions at infinity are considered in (Sato, 1999, Chapter 5, Sections 25 and 26). Our survey is by no way complete, though.

Theorems 4.1.1–4.1.3 are proved in (Yakymiv, 1987b).

Let us turn to proving Theorems 4.1.1–4.1.3 and their corollaries. We introduce

$$F(t) = \mathbf{P}\{\xi \le t\}, \qquad T(t) = 1 - F(t),$$
$$q_1(t) = -\frac{d}{dt}q(t), \qquad Q(t) = q(0) - q(t)$$

(provided that $q(0) < \infty$)

$$r(t) = \begin{cases} q(t), & t > 0,\\ 0, & t = 0,\end{cases}$$
$$I_n = \int_0^\infty t^n q(t)\,dt, \qquad n = 0, 1, 2, \ldots,$$
$$h(t) = \begin{cases} q(t)/t^2, & I_0 < \infty,\\ q^2(t)/t & \text{otherwise;}\end{cases}$$

let $f^{(n)}(t)$ denote the nth derivative of a function $f(t)$, while $f^{(0)}(t) = f(t)$; let $r_n(t)$ be the n-fold convolution of a function $r(t)$ with itself: $r_1(t) = r(t)$, and for $n > 1$

$$r_n(t) = \int_0^t r_{n-1}(t-u)\,dr(u), \qquad t \geq 0.$$

The Laplace transform and the Laplace–Stieltjes transform of a function f

$$\int_0^\infty e^{-t\lambda} f(t)\,dt, \qquad \int_0^\infty e^{-t\lambda}\,df(t)$$

are denoted by $\widehat{f}(\lambda)$ and $\widetilde{f}(\lambda)$ respectively.

Before proving Theorems 4.1.1–4.1.3, we formulate and prove a series of lemmas.

LEMMA 4.1.1. *Let there exist $a > 0$ such that $q(t) = q(a)$, $t \in [0, a]$, and $q(t) \in C^1[a, \infty)$. Then for all $n \in \mathbf{N}$*

$$r_n(t) = (q(a))^n, \quad t \in (0, a], \quad r_n(t) \in C^1[a, \infty),$$

and for all $t \in [a, \infty)$

$$r_n'(t) = q(0) r_{n-1}'(t) - q_1(t) r_{n-1}(t) - \int_0^t r_{n-1}'(t-u)(q_1(u) - q_1(t))\,du.$$

LEMMA 4.1.2. *Let* (4.1.1) *hold and $q(0) < \infty$. Then there exists a constant $c < \infty$ such that for all $t \geq 0$ and $n \in N$*

$$|r_n(t)| \leq c^n q(t). \tag{4.1.2}$$

LEMMA 4.1.3. *Let* (4.1.1) *hold, and let $q''(t)$ do not increase for sufficiently large t. Then $t^2 q''(t)$ and $t q'(t)$ are $O(q(t))$ as $t \to \infty$.*

LEMMA 4.1.4. *For some $a > 0$, let $q(t) = q(a)$, $t \in (0, a]$, $q(t) \in C^1[a, \infty)$, let $q'(t)$ be concave on the set $[a, \infty)$, and let one of the assumption below be fulfilled:*

(A) *$I_0 = \infty$ and $q(t)$ is regularly varying at infinity;*

(B) *$I_0 < \infty$ and* (4.1.1) *holds.*

Then there exists a constant $b < \infty$ such that for all $t > 0$, $n > 1$

$$|r_n'(t)| \leq b^n h(t). \tag{4.1.3}$$

LEMMA 4.1.5. *Let* (4.1.1) *hold. Then for all $i, j \geq 0$ as $\lambda \downarrow 0$*

$$\widehat{q}^{(i)}(\lambda)\widehat{q}^{(j)}(\lambda) = O(|\widehat{q}^{(i+j+1)}(\lambda)|),$$

and the symbol O in the last expression can be replaced by o if $I_{i+j+1} = \infty$.

LEMMA 4.1.6. *Let* (4.1.1) *hold, and let an n-tuple of non-negative integers* $i_1, \dots, i_n$ *obey the relation*

$$\sum_{j=1}^{n} j i_j = n.$$

Then, as $\lambda \downarrow 0$,

$$\prod_{j=1}^{n} ((\lambda \widehat{q}(\lambda))^{(j)})^{i_j} = O(|\widehat{q}^{(n-1)}(\lambda)|),$$

and the symbol O in the last expression can be replaced by o if $i_n = 0$ *and* $I_{n-1} = \infty$.

LEMMA 4.1.7. *Let* (4.1.1) *hold and* $I_0 < \infty$. *Then for all* $i, j \in \mathbf{N}$

$$\widehat{q}^{(i)}(\lambda)\widehat{q}^{(j)}(\lambda) = O(|\widehat{q}^{(i+j)}(\lambda)|), \qquad \lambda \downarrow 0,$$

and the symbol O in the last expression can be replaced by o if $I_{i+j} = \infty$.

PROOF OF LEMMA 4.1.1. For $n = 1$, the lemma is,obviously, true. Let the lemma be true for $n-1$, $n > 1$. Since

$$r_1(t) = q(0), \qquad r_{n-1}(0) = (q(0))^{n-1}$$

for $t \in (0, a]$, we see that

$$r_n(t) = \int_0^t r_{n-1}(t-u)\, dr_1(u) = r_{n-1}(t)q(0) = (q(0))^n$$

for $t \in (0, a]$. For $t > a$ we find that

$$\begin{aligned} r_n'(t) &= \frac{d}{dt}\left(q(0)r_{n-1}(t) - \int_0^t r_{n-1}(t-u)q_1(u)\, du\right) \\ &= q(o)r_{n-1}'(t) - q_1(t)r_{n-1}(a) - \int_a^t r_{n-1}'(t-u)q_1(u)\, du, \end{aligned}$$

hence,

$$\begin{aligned} r_n'(t) &= q(0)r_{n-1}'(t) - q_1(t)r_{n-1}(t) + (r_{n-1}(t) - r_{n-1}(a))q_1(t) \\ &\quad - \int_a^t r_{n-1}'(t-u)q_1(u)\, du \\ &= q(0)r_{n-1}'(t) - q_1(t)r_{n-1}(t) - \int_a^t r_{n-1}'(t-u)(q_1(u) - q_1(t))\, du. \end{aligned}$$

The lemma is proved. □

PROOF OF LEMMA 4.1.2. We observe that

$$\begin{aligned} \int_0^t q(t-u)\, dQ(u) &= \int_0^{t/2} q(t-u)\, dQ(u) + \int_{t/2}^t q(t-u)\, dQ(u) \\ &\le q(t/2)q(0) + q(0)(q(t/2) - q(t)) \le 2q(0)q(t/2). \end{aligned}$$

We set

$$c = \max(1, q(0) + c_1),$$

where

$$c_1 = \sup_{t\geq 0} \int_0^t q(t-u)\, dQ(u)/q(t).$$

In view of the above bounds, $c < \infty$. Let us prove relation (4.1.2) by induction. For $n = 1$ it is obviously true. If $n > 1$ and (4.1.2) holds true for $n - 1$, then

$$\begin{aligned} |r_n(t)| &= \left| q(0) r_{n-1}(t) - \int_0^t r_{n-1}(t-u)\, dQ(u) \right| \\ &\leq q(0) c^{n-1} q(t) + c^{n-1} \int_0^t q(t-u)\, dQ(u) \\ &\leq c^{n-1}(q(0) + c_1) q(t) = c^n q(t) \end{aligned}$$

for all $t \geq 0$. The lemma is proved. □

PROOF OF LEMMA 4.1.3. The lemma follows from the inequalities

$$q(t) \geq \int_t^{2t} q_1(u)\, du \geq t q_1(2t) \geq t \int_{2t}^{3t} q''(u)\, du \geq t^2 q''(3t)$$

and the fact that

$$q(t) \asymp q(2t) \asymp q(3t)$$

as $t \to \infty$. □

PROOF OF LEMMA 4.1.4. The constants defined by the formulas

$$\begin{aligned} c_2 &= \sup_{t\geq a} \left(\int_0^{t-a} h(t-u)(q_1(u) - q_1(t))\, du / h(t) \right), \\ c_3 &= \sup_{t\geq a} (|r_2'(t)| / h(t)), \\ c_4 &= \sup_{t\geq a} (q(t) q_1(t) / h(t)) \end{aligned}$$

are finite. It is easy to see, indeed, that for $t > 2a$

$$\begin{aligned} 0 &\leq \int_0^{t/2} h(t-u)(q_1(u) - q_1(t))\, du \\ &\leq h(t/2) \int_0^{t/2} q_1(u)\, du = O(h(t)), \qquad t \to \infty. \end{aligned} \tag{4.1.4}$$

Further, by virtue of Lemma 4.1.3,

$$\begin{aligned} \int_{t/2}^{t-a} h(t-u)(q_1(u) - q_1(t))\, du &\leq q''(t/2) \int_0^{t/2} u h(u)\, du \\ &= O(q(t)/t^2) \int_a^t u h(u)\, du = O(h(t)), \end{aligned} \tag{4.1.5}$$

because in case B, as $t \to \infty$,

$$I(t) = \int_a^t uh(u)\,du = \int_a^t q^2(u)\,du \le q(0) \int_a^t q(u)\,du = O(tq(t)),$$

and in case A $I(t) = O(1)$ as $t \to \infty$. Finiteness of c_2 follows from (4.1.4) and (4.1.5). Next, for $t > 2a$

$$\begin{aligned} r_2'(t) &= \frac{d}{dt}\left(q(o)q(t) - \int_a^t q_1(u)q(t-u)\,du\right) \\ &= -2q(0)q_1(t) + \int_0^t q_1(t-u)q_1(u)\,du, \end{aligned}$$

hence it follows that

$$\begin{aligned} r_2'(t) &= 2\left(\int_0^{t/2} q_1(u)q_1(t-u)\,du - q_1(t)\int_0^\infty q_1(u)\,du\right) \\ &= 2\left(\int_0^{t/2} q_1(u)(q_1(t-u) - q_1(t))\,du - q_1(t)q(t/2)\right). \end{aligned}$$

From the obtained relation and Lemma 4.1.3 it follows that

$$r_2'(t) \ge -q_1(t)q(t/2) + O(q^2(t)/t) = O(h(t)), \tag{4.1.6}$$

$$r_2'(t) \le 2q''(t/2)\int_0^{t/2} uq_1(u)\,du = O\left(\frac{q(t)}{t^2}\int_0^t q(u)\,du\right) = O(h(t)) \tag{4.1.7}$$

as $t \to \infty$. Finiteness of c_3 follows from (4.1.6) and (4.1.7). Next, by virtue of Lemma 4.1.3,

$$q(t)q_1(t) = O(q^2(t)/t), \qquad t \to \infty,$$

which implies that $c_4 < \infty$. Let

$$b = \max(c, q(o) + c_2 + c_4, \sqrt{c_3}).$$

Let us prove that inequality (4.1.3) holds true for all $n > 1$. For $n = 2$ it follows from finiteness of c_3 and the inequality $c_3^2 \le b$. If $n > 2$ and (4.1.3) is true for $n-1$, then, by virtue of Lemmas 4.1.1 and 4.1.2,

$$\begin{aligned} |r_n'(t)| &= \left| q(0)r_{n-1}'(t) - q_1(t)r_{n-1}(t) - \int_0^t r_{n-1}'(t-u)(q_1(u) - q_1(t))\,du \right| \\ &\le q(0)|r_{n-1}'(t)| + c^{n-1}q_1(t)q(t) + \int_0^t |r_{n-1}'(t-u)|(q_1(u) - q_1(t))\,du. \end{aligned}$$

Hence, by the induction assumption, we obtain

$$|r_n'(t)| \le q(0)b^{n-1}h(t) + c^{n-1}c_4h(t) + b^{n-1}\int_0^t h(t-u)(q_1(u) - q_1(t))\,du,$$

or

$$|r_n'(t)| \le h(t)(q(0)b^{n-1} + c_4c^{n-1} + c_2b^{n-1}) \le h(t)b^{n-1}(q(0) + c_4 + c_2) \le b^n h(t).$$

The lemma is thus proved. □

PROOF OF LEMMA 4.1.5. We observe that

$$\widehat{q}^{(i)}(\lambda)\widehat{q}^{(j)}(\lambda) = (-1)^{i+j}\widehat{I}(\lambda),$$

where

$$I(x) = \int_0^x u^i q(u)(x-u)^j q(x-u)\,du, \qquad x > 0.$$

By virtue of (4.1.1), the inequalities

$$\begin{aligned}\int_0^{t/2} u^i q(u)(t-u)^j q(t-u)\,du &\le t^j q(t/2)\int_0^t u^i q(u)\,du \\ &= t^{i+j+1}q(t/2)\left(\int_0^t u^i q(u)\,du/t^{i+1}\right) \\ &= t^{i+j+1}q(t)o(1)\end{aligned}$$

are true as $t \to \infty$. Using the same estimates for the integral from $t/2$ to t, we conclude that

$$I(t) = o(t^{i+j+1}q(t)), \qquad t \to \infty,$$

which proves the lemma. □

PROOF OF LEMMA 4.1.6. We observe that for any $j \in \mathbf{N}$

$$(\lambda\widehat{q}(\lambda))^{(j)} = j\widehat{q}^{(j-1)}(\lambda) + \lambda\widehat{q}^{(j)}(\lambda) = O(|\widehat{q}^{(j-1)}(\lambda)|), \qquad \lambda \downarrow 0.$$

It remains to make use of Lemma 4.1.5. □

PROOF OF LEMMA 4.1.7. The bound

$$\int_0^{t/2} u^i q(u)(t-u)^j q(t-u)\,du \le t^{i+j}q(t/2)\left(\int_0^t u^i q(u)\,du/t^i\right) \tag{4.1.8}$$

is true. Besides, under the hypotheses of the lemma

$$\int_0^t u^i q(u)\,du = o(t^i), \qquad t \to \infty. \tag{4.1.9}$$

It is easily seen, indeed, that for any $\varepsilon \in (0,1)$

$$\begin{aligned}\int_0^t \left(\frac{u}{t}\right)^i q(u)\,du &\le \int_0^{t\varepsilon} \left(\frac{u}{t}\right)^i q(u)\,du + \int_{t\varepsilon}^t \left(\frac{u}{t}\right)^i q(u)\,du \\ &\le \varepsilon^i \int_0^\infty q(u)\,du + \int_{t\varepsilon}^\infty q(u)\,du.\end{aligned}$$

Since $I_0 < \infty$, hence it follows that

$$\limsup_{t\to\infty} \int_0^t \left(\frac{u}{t}\right)^i q(u)\,du \le \varepsilon^i \int_0^\infty q(u)\,du.$$

Relation (4.1.9) is true because ε is arbitrary. From (4.1.1), (4.1.8), and (4.1.9) it follows that

$$\int_0^{t/2} u^i q(u)(t-u)^j q(t-u)\,du = o(t^{i+j}q(t)), \qquad t\to\infty.$$

Using the same estimates for the integral from $t/2$ to t, we conclude that

$$\int_0^{t} u^i q(u)(t-u)^j q(t-u)\,du = o(t^{i+j}q(t)), \qquad t\to\infty.$$

The last asymptotic relation proves the lemma. □

PROOF OF THEOREM 4.1.1. We set

$$\Phi(t) = (1-\exp(-t))/t, \qquad t>0,$$
$$\Psi(\lambda) = \Phi(\lambda\widehat{q}(\lambda)), \qquad \lambda>0.$$

Without loss of generality (Feller, 1966, Section XVII.4), we set $\xi \ge 0$ and $q(0) < \infty$. Assuming that the location parameter is zero, we obtain

$$\mathbf{E}e^{-\lambda\xi} = \exp\left(-\int_0^\infty (1-e^{-\lambda x})G(dx)\right), \qquad \forall\lambda>0,$$

hence

$$\widehat{T}(\lambda) = (1-\exp(-\lambda\widehat{q}(\lambda)))/\lambda = \Phi(\lambda\widehat{q}(\lambda))\widehat{q}(\lambda) = \Psi(\lambda)\widehat{q}(\lambda). \tag{4.1.10}$$

From the definition of the function $\Phi(x)$ it follows that for any positive integer m

$$\Phi^{(m)}(t) \to (-1)^m/(m+1), \qquad t\downarrow 0. \tag{4.1.11}$$

Furthermore, for $m\in\mathbf{N}$

$$\widehat{T}(\lambda) = \sum_{k=0}^m \binom{m}{k}\Psi^{(k)}(\lambda)\widehat{q}^{(m-k)}(\lambda). \tag{4.1.12}$$

Using formula (0.430) in (Gradshteyn, Ryzhik, 1980) for the mth derivative of a composite function, we obtain

$$\Psi^{(m)}(\lambda) = \sum_{i_1,\dots,i_m} \Phi^{(k)}(\lambda\widehat{q}(\lambda))C(i_1,\dots,i_m)\prod_{j=1}^m ((\lambda\widehat{q}(\lambda))^{(j)})^{i_j}, \tag{4.1.13}$$

where the summation is over all ordered sets of non-negative integers $i_1,\dots,i_m$ which obey the relation

$$\sum_{j=1}^m j i_j = m,$$

and

$$k = \sum_{j=1}^{m} i_j,$$

while $C(i_1, \dots, i_m)$ are some constants. Hence, in view of (4.1.11) and Lemma 4.1.6, we obtain

$$\Psi^{(m)}(\lambda) = O(|\widehat{q}^{(m-1)}(\lambda)|), \qquad \lambda \downarrow 0, \tag{4.1.14}$$

for any $m \in \mathbf{N}$. We choose $M \in \mathbf{N}$ so that $I_M = \infty$. We fix an arbitrary $n \geq M$. From (4.1.12) and (4.1.14) it follows that

$$\widehat{T}^{(n)}(\lambda) = \Psi(\lambda)\widehat{q}^{(n)}(\lambda) + O\left(\sum_{k=1}^{n} |\widehat{q}^{(k-1)}(\lambda)\widehat{q}^{(n-k)}(\lambda)|\right)$$

as $\lambda \downarrow 0$. Therefore, by virtue of Lemma 4.1.5 and relation (4.1.11),

$$\widehat{T}^{(n)}(\lambda) = \Psi(\lambda)\widehat{q}^{(n)}(\lambda) + o(|\widehat{q}^{(n)}(\lambda)|) = (1 + o(1))\widehat{q}^{(n)}(\lambda), \qquad \lambda \downarrow 0.$$

In order to prove the theorem, it remains to use Theorem 1.6.2. □

PROOF OF THEOREM 4.1.2. First we assume that $G((-\infty, a]) = 0$ for some $a > 0$. Then (4.1.10) holds. We set

$$\rho(\lambda) = \widehat{T}(\lambda) - \widehat{q}(\lambda),$$
$$\varphi(\lambda) = (1 - \exp(-\lambda) - \lambda)/\lambda,$$

and rewrite (4.1.10) as follows:

$$\rho(\lambda) = \psi(\lambda)\widehat{q}(\lambda), \tag{4.1.15}$$

where

$$\psi(\lambda) = \varphi(\lambda\widehat{q}(\lambda), \qquad \lambda > 0. \tag{4.1.16}$$

For $\varphi(t)$, the relations

$$\varphi(t) = -t/2 + o(t),$$
$$\varphi^{(m)}(t) = \frac{(-1)^m}{m+1} + o(1), \qquad t \to 0, \tag{4.1.17}$$

are true for fixed $m \in \mathbf{N}$. Therefore, as $\lambda \downarrow 0$, (4.1.16) and (4.1.17) yield

$$\psi(\lambda) = -\frac{\lambda A}{2}(1 + o(1)), \tag{4.1.18}$$

$$\psi'(\lambda) = \varphi'(\lambda\widehat{q}(\lambda))(\lambda\widehat{q}(\lambda))' = -\frac{A}{2}(1 + o(1)), \tag{4.1.19}$$

where $A = \mathsf{M}\xi$. Making use of formula (0.430) in (Gradshteyn, Ryzhik, 1980) for the mth derivative of a composite function, we see that

$$\psi^{(m)}(\lambda) = \sum_{i_1,\dots,i_m} \varphi^{(k)}(\lambda\widehat{q}(\lambda))C(i_1,\dots,i_m)\prod_{j=1}^{m}((\lambda\widehat{q}(\lambda))^{(j)})^{i_j}, \tag{4.1.20}$$

where the constants $C(i_1,\dots,i_m)$ and the summation domain are the same as in formula (4.1.13). By virtue of (4.1.17) and Lemma 4.1.6, hence we find that for any $m \in \mathbf{N}$

$$\psi^{(m)}(\lambda) = O(|\widehat{q}^{(m-1)}(\lambda)|), \qquad \lambda \downarrow 0. \tag{4.1.21}$$

Let $M \in \mathbf{N}$ be chosen so that $I_{M-1} = \infty$. We fix an arbitrary $n \geq M$. Setting $m = n$ in (4.1.20), making use of (4.1.17) and Lemma 4.1.6, we conclude that, as $\lambda \downarrow 0$,

$$\psi(n)(\lambda) = \varphi'(\lambda\widehat{q}(\lambda))(\lambda\widehat{q}(\lambda))^{(n)} + o(|\widehat{q}^{(n-1)}(\lambda)|), \tag{4.1.22}$$

because $C(0,\dots,0,1) = 1$. By virtue of (4.1.21) and Lemma 4.1.7 we see that, as $\lambda \downarrow 0$,

$$\begin{aligned}\rho^{(n)}(\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\varphi^{(k)}(\lambda)\widehat{q}^{(n-k)}(\lambda)\\ &= \psi(\lambda)\widehat{q}^{(n)}(\lambda) + n\psi'(\lambda)\widehat{q}^{(n-1)}(\lambda) + \psi^{(n)}(\lambda)\widehat{q}(\lambda) + o(|\widehat{q}^{(n-1)}(\lambda)|),\end{aligned}$$

which, in view of relations (4.1.18), (4.1.19) and (4.1.22), yields

$$\begin{aligned}\rho^{(n)}(\lambda) &= -\frac{\lambda A}{2}\widehat{q}^{(n)}(\lambda)(1+o(1)) - \frac{A}{2}n\widehat{q}^{(n-1)}(\lambda) - \frac{A}{2}(\lambda\widehat{q}(\lambda))^{(n)} + o(|\widehat{q}^{(n-1)}(\lambda)|)\\ &= -A(\lambda\widehat{q}(\lambda))^{(n)} + o(|\widehat{q}^{(n-1)}(\lambda)|), \qquad \lambda \downarrow 0.\end{aligned} \tag{4.1.23}$$

We set

$$f(t) = T(t) - q(t) - \mathbf{E}\xi q'(t), \quad g(t) = q(t)/(t+1), \quad t \geq 0.$$

From (4.1.23) it follows that

$$\widehat{f}(\lambda) = o(|\widehat{g}^{(n)}(\lambda)|), \qquad \lambda \downarrow 0. \tag{4.1.24}$$

From the equalities

$$\begin{aligned}\widetilde{F}(\lambda) = \exp(-\lambda\widehat{q}(\lambda)) &= \sum_{k\geq 0}(-\lambda\widehat{q}(\lambda))^k/k!\\ &= \sum_{k\geq 0}(-\widetilde{r}(\lambda))^k/k! = \sum_{k\geq 0}(-1)^k\widetilde{r}_k(\lambda)/k!\end{aligned}$$

it follows that

$$T(t) - q(t) = \sum_{k\geq 2}\frac{r_k(t)}{k!}(-1)^{k+1}. \tag{4.1.25}$$

By virtue of Lemmas 4.1.3 and 4.1.4, hence we obtain

$$f'(t) = O(g(t)/t), \qquad t \to \infty. \tag{4.1.26}$$

Beginning with relations (4.1.24) and (4.1.25), with the use of Theorem 1.6.3, we see that Theorem 4.1.2 is true in the special case just considered.

In the general case, the distribution of ξ admits the representation (Feller, 1966, Section XVII.4 (d)) as the distribution of the sum of three independent random variables τ, $-\tau_1$, and τ_2 such that for any $m > 0$

$$\mathsf{P}\{|\tau_2| > t\} = o(t^{-m}), \qquad t \to \infty, \tag{4.1.27}$$

whereas the random variables $\tau \geq 0$ and $\tau_1 \geq 0$ have the spectral functions $q(\max(t, a))$ and $p(\max(t, a))$ respectively. We set $\nu = \tau_2 - \tau_1$. Since $\tau_1 \geq 0$, from (4.1.26) it follows that for any $m \in \mathbf{N}$

$$\mathsf{P}\{\nu > t\} = o(t^{-m}), \qquad t \to \infty. \tag{4.1.28}$$

By the proved above, as $t \to \infty$,

$$\mathsf{P}\{\tau > t\} = q(t) + \mathsf{E}\tau q_1(t) + O(q(t)/t). \tag{4.1.29}$$

We set

$$R(t) = \mathsf{P}\{\tau > t\}, \qquad H(t) = \mathsf{P}\{\nu \leq t\}.$$

We observe that

$$\begin{aligned} \mathsf{P}\{\tau + \nu > t\} - \mathsf{P}\{\tau > t\} + \mathsf{E}R(t - \nu) - R(t) &= \int_{-\infty}^{\infty} (R(t-u) - R(t))\, dH(u) \\ &= T_1 + T_2 + T_3, \end{aligned} \tag{4.1.30}$$

where T_1, T_2, T_3 are the integrals over the sets $(-\infty, -\sqrt{t})$, $[-\sqrt{t}, \sqrt{t}]$, $(\sqrt{t}, \infty)$ respectively. By (4.1.29),

$$\begin{aligned} T_1 &= \int_{-\infty}^{-\sqrt{t}} (R(t-u) - R(t))\, dH(u) \\ &= \int_{-\infty}^{-\sqrt{t}} (q(t-u) - q(t))\, dH(u) + o(q(t)/t) \end{aligned}$$

as $t \to \infty$. Since for $u < 0$ the difference $q(t) - q(t-u)$ is positive and does not exceed $|u| q_1(t)$, from the estimate obtained it follows that $T_1 = o(q(t)/t)$ as $t \to \infty$. Further,

$$\begin{aligned} |T_3| &= \left| \int_{\sqrt{t}}^{\infty} (R(t-u) - R(t)\, dH(u) \right| \leq \int_{\sqrt{t}}^{\infty} dH(u) \\ &= \mathsf{P}\{\nu > t\} = o(q(t)/t), \qquad t \to \infty, \end{aligned}$$

by virtue of (4.1.28). In view of (4.1.29), T_2 is estimated as follows:

$$T_2 = \int_{-\sqrt{t}}^{\sqrt{t}} (q(t-u) - q(t))\, dH(u) + \mathbf{E}\tau \int_{-\sqrt{t}}^{\sqrt{t}} (q_1(t-u) - q_1(t))\, dH(u) + o(q(t)/t)$$

as $t \to \infty$. We observe that

$$\int_{-\sqrt{t}}^{\sqrt{t}} |q_1(t-u) - q_1(t)|\, dH(u) \le q''(t - \sqrt{t}) \int_{-\infty}^{\infty} |u|\, dH(u).$$

The last expression, by virtue of Lemma 4.1.3, is $o(q(t)/t)$ as $t \to \infty$. Therefore, by the mean value theorem, for some $\theta \in [-1, 1]$ depending on t,

$$\begin{aligned} T_2 &= q_1(t + \theta\sqrt{t}) \int_{-\sqrt{t}}^{\sqrt{t}} u\, dH(u) + o(q(t)/t) \\ &= q_1(t + \theta\sqrt{t})\mathbf{E}\nu + o(q(t)/t), \qquad t \to \infty. \end{aligned}$$

Thus, as $t \to \infty$

$$T_2 = q_1(t)\mathbf{E}\nu + o(q(t)/t),$$

because by Lemma 4.1.3

$$\begin{aligned} |q_1(t + \theta\sqrt{t}) - q_1(t)| &\le \sqrt{t} q''(t - \sqrt{t}) \\ &= O(\sqrt{t} q(t)/t^2) = o(q(t)/t), \qquad t \to \infty. \end{aligned}$$

From (4.1.29) and (4.1.30), in view of the above estimates of T_1, T_2, T_3, we obtain

$$\mathbf{P}\{\tau + \nu > t\} = q(t) + \mathbf{E}(\tau + \nu) q_1(t) + o(q(t)/t), \qquad t \to \infty.$$

The theorem is thus proved. □

PROOF OF THEOREM 4.1.3. Here we use the notation introduced in the proof of Theorem 4.1.2. In the case where $G((-\infty, a]) = 0$, from relations (4.1.15)–(4.1.17) and (4.1.19) we obtain

$$\begin{aligned} \rho'(\lambda) &= \varphi'(\lambda\widehat{q}(\lambda))(\lambda\widehat{q}(\lambda))'\widehat{q}(\lambda) + \varphi(\lambda\widehat{q}(\lambda))\widehat{q}'(\lambda) \\ &= -\frac{1}{2}((\lambda\widehat{q}(\lambda))'\widehat{q}(\lambda)(1 + o(1)) - \lambda\widehat{q}(\lambda)\widehat{q}'(\lambda)(1 + o(1))) \\ &= \lambda\widehat{q}(\lambda)\widehat{q}'(\lambda) - \frac{1}{2}\widehat{q}^2(\lambda) + o(\widehat{q}^2(\lambda)), \qquad \lambda \downarrow 0. \end{aligned}$$

Hence it follows that for $t = 1/\lambda$

$$\begin{aligned} \rho'(\lambda) &= -\Gamma(1-\alpha)\Gamma(2-\alpha)t^2 q^2(t) + \frac{1}{2}\Gamma^2(1-\alpha)t^2 q^2(t) + o(t^2 q^2(t)) \\ &= \frac{1}{2}\Gamma^2(1-\alpha)(1-2\alpha)t^2 q^2(t) + o(t^2 q^2(t)) \\ &= \gamma\widehat{g}'(\lambda) + o(|\widehat{g}'(\lambda)t|), \qquad \lambda \downarrow 0, \end{aligned}$$

where

$$g(t) = q^2(t), \qquad \gamma = \frac{\Gamma(1-\alpha)(1-2\alpha)}{2\Gamma(2-2\alpha)}.$$

So for the function $f(t) = T(t) - q(t) - \gamma g(t)$ we obtain

$$\widehat{f}'(\lambda) = o(|\widehat{g}'(\lambda)|), \qquad \lambda \downarrow 0.$$

From representation (4.1.25), Lemmas 4.1.3 and 4.1.4 it follows that

$$f'(t) = O(g(t)/t), \qquad t \to \infty.$$

In view of Theorem 1.6.3 and Remark 1.6.1, hence it follows that the theorem is valid in the just considered special case. Next, we represent the integral in (4.1.30) as the sum of the integrals J_1, J_2, J_3 over the intervals $(-\infty, 0)$, $[0, \sqrt{t}]$, $(\sqrt{t}, \infty)$ respectively. The integral J_3 is estimated in the same way as T_3. By virtue of the inequality

$$0 \le q^2(t-u) - q^2(t) \le q(t-\sqrt{t})(q(t-u) - q(t)),$$

which holds true for $t \ge 1$ and $u \le \sqrt{t}$, we obtain

$$\begin{aligned} J_2 &= \int_0^{\sqrt{t}} (q(t-u) - q(t)) dH(u) + o(q^2(t)) \\ &= O\left(q_1(t) \int_0^{\sqrt{t}} u\, dH(u) \right) + o(q^2(t)) = o(q^2(t)), \qquad t \to \infty. \end{aligned}$$

Besides,

$$\begin{aligned} J_1 &= \int_{-\infty}^{0} (q(t-u) - q(t))\, dH(u) + o(q^2(t)) \\ &= -\int_0^{\infty} q_1(t+u) H(-u) du + o(q^2(t)). \end{aligned}$$

We set

$$Q_1(t) = \mathbf{P}\{\tau_1 \ge t\}.$$

Since

$$H(-t) = Q_1(t) + o(t^{-2}), \qquad t \to \infty,$$

we obtain

$$J_1 = -\int_0^{\infty} q_1(t+u) Q_1(u)\, du + o(q^2(t)), \qquad t \to \infty.$$

The last expression is $o(q^2(t))$ as $t \to \infty$, provided that $\mathbf{E}\tau_1 < \infty$. Let us consider the case where $\mathbf{E}\tau_1 = \infty$. First,

$$\begin{aligned} \int_0^t q_1(t+u) Q_1(u)\, du &\le q_1(t) \int_0^t Q_1(u)\, du \\ &= q_1(t) \int_0^t p(u)\, du (1 + o(1)), \qquad t \to \infty. \end{aligned}$$

Second,

$$\int_t^\infty q_1(t+u)Q_1(u)\,du \le Q_1(t)\int_t^\infty q_1(t+u)\,du$$
$$= Q_1(t)q(2t) \le \frac{q(2t)}{t}\int_0^t Q_1(u)\,du.$$

Therefore, as $t \to \infty$,

$$J_1 = O\left(\frac{q(t)}{t}\int_0^t p(u)\,du\right) + o(q^2(t)).$$

These estimates prove the theorem. □

Let us give an example where $f(t) = \mathbf{P}\{\xi > t\} \overset{w}{\sim} q(t)$ but $f(t) \nsim q(t)$ as $t \to \infty$. We set $q(t) = 1/t$ for $t \in (0,1)$, $q(t) = 2^{-n}$ for $t \in [2^n, 2^{n+1})$, $n \ge 0$. It is clear that $q(t)/q(2t) \le 2$ for $t \ge 1$, so (4.1.1) holds. Therefore, $f(t) \overset{w}{\sim} q(t)$ as $t \to \infty$. Assume that $f(t) \sim q(t)$ as $t \to \infty$. Then there exists $b \ge 1$ such that for $t \ge b$

$$\frac{3}{4}q(t) \le f(t) \le \frac{5}{4}q(t). \tag{4.1.31}$$

We choose n in such a way that $2^{n-1} \ge b$ and set $\tau = 2^n$. By (4.1.31), we obtain

$$\frac{3}{4}2^{-n} \le f(\tau) \le \frac{5}{4}2^{-n}. \tag{4.1.32}$$

For $\varepsilon \in (0, 2^{-n})$, by (4.1.31),

$$\frac{3}{4}2^{-(n-1)} \le f(\tau-\varepsilon) \le \frac{5}{4}2^{-(n-1)}. \tag{4.1.33}$$

From (4.1.32) and (4.1.33) it follows that

$$f(\tau-\varepsilon) - f(\tau) \ge \frac{3}{4}2^{-(n-1)} - \frac{5}{4}2^{-n} = \frac{1}{4}2^{-n}.$$

The last relation contradicts the continuity of the function f at the point τ. In other words, $f(t) \nsim q(t)$ as $t \to \infty$.

Tauberian theorems are applied to analysing asymptotic properties of stable, particularly Gaussian, processes and convergence to them in (Kasahara, Kosugi, 2002; Kasahara *et al.*, 1999; Kono, Ogawa, 1999; Geluk, Peng, 2000; Feigin, Yashchin, 1983; Li, Shao, 2001; Broniatowski, Fuchs, 1995; Janssen, 1985).

4.2. Asymptotic behaviour of a density at infinity

Let a random variable ξ have an infinitely divisible distribution with Lévy spectral measure $G(dx)$, that is, let the representation of the characteristic function of ξ given in Section 4.1 take place. In this section we find the asymptotic behaviour at infinity of the function

$$f(t) = \frac{d}{dt}\mathbf{P}\{\xi \le t\}$$

(under the assumption that the last derivative exists for $t > 0$). The following two limit theorems are true.

THEOREM 4.2.1. *Let the measure $G(dx)$ be bounded and have a density $g(x)$ which is continuous on $[0, \infty)$, and let there exist $\alpha > 0$ such that the function $b(t) = tg(t)$ does not increase for $t \geq \alpha$, while*

$$\limsup_{t\to\infty} \frac{g(t)}{g(2t)} < \infty, \tag{4.2.1}$$

that is, $g(t)$ is dominatedly varying at infinity (see Remark 1.6.1) and for any $\lambda > 1$ let

$$\limsup_{t\to\infty} \frac{r(\lambda t)}{r(t)} < 1, \tag{4.2.2}$$

where $r(t) = G([t, \infty))$, $t > 0$. Then, as $t \to \infty$,

$$f(t) \overset{w}{\sim} g(t). \tag{4.2.3}$$

The definition of weak equivalence of functions is given in Section 1.6.

THEOREM 4.2.2. *Let the measure G admit the representation $G = G_A + G_B$, where G_A is its absolutely continuous part, and let $g(t)$ be the density of the measure G_A. We assume that for all $n \in \mathbf{N}$*

$$\int_1^\infty t^n G_B(dt) < \infty, \tag{4.2.4}$$

there exist $\varepsilon > 0$ and $\alpha > \varepsilon$ such that

$$g(t) \geq t^{-1}, \qquad \forall t \in (0, \varepsilon], \tag{4.2.5}$$

the function $b(t) = tg(t)$ is monotone and continuous for $t \geq \alpha$, relations (4.2.1) *and* (4.2.2) *are true. Then* (4.2.3) *holds.*

Theorems 4.2.1 and 4.2.2 are obtained in (Yakymiv, 2002). The first result in this direction is given in (Yakymiv, 1990b).

Let us discuss condition (4.2.2) of Theorem 4.2.1. It is well known in the theory of dominatedly varying functions (Seneta, 1976, Section A.3, Theorem A.5) that (4.2.1) holds if and only if there exist positive c, x_0 and real β such that the inequality

$$\frac{g(y)}{g(x)} \leq c \left(\frac{y}{x}\right)^\beta \tag{4.2.6}$$

holds for $y \geq x \geq x_0$. So, in order for (4.2.2) to be true, it is sufficient that $\beta < -1$ in (4.2.6). It is easily seen, indeed, that (4.2.2) holds if and only if

$$\liminf_{t\to\infty} \frac{\int_t^{\lambda t} g(x)\,dx}{\int_t^\infty g(x)\,dx} > 0,$$

which is equivalent to the relation

$$\limsup_{t\to\infty} \frac{\int_t^\infty g(x)\,dx}{\int_t^{\lambda t} g(x)\,dx} < \infty. \tag{4.2.7}$$

Since $g(x)$ is dominatedly varying, (4.2.7) is true if and only if

$$\limsup_{t\to\infty} \frac{\int_t^\infty g(x)\,dx}{t g(t)} < \infty.$$

Changing the variable $x = ut$, we arrive at the relation

$$\limsup_{t\to\infty} \int_1^\infty \frac{g(ut)}{g(t)}\,du < \infty. \tag{4.2.8}$$

It remains to say that (4.2.8), and hence (4.2.2), is true if inequality (4.2.6) holds for some $\beta < -1$.

We give one more sufficient condition for (4.2.2). Since $g(x)$ is dominatedly varying, for some $a > 0$ and measurable bounded functions $\eta(x)$, $\varepsilon(x)$ on $[a, \infty)$ the representation

$$g(x) = \exp\left(\eta(x) + \int_a^x \frac{\varepsilon(t)}{t}\,dt\right) \tag{4.2.9}$$

is true for all $x \geq a$ (Seneta, 1976, Section A.1, Theorem A.1). For (4.2.6) to be true, it suffices that

$$\sup_{x\geq a} \varepsilon(x) = \beta$$

in (4.2.9). So, (4.2.2) holds if

$$\sup_{x\geq a} \varepsilon(x) = \beta < -1.$$

Let

$$F(t) = \mathbf{P}\{\xi \leq t\}.$$

As before, let

$$\widetilde{H}(\lambda) = \int_0^\infty e^{-t\lambda}\,dH(t), \qquad \lambda \geq 0,$$

stand for the Laplace–Stieltjes transform of a function $H(t)$.

We formulate four auxiliary assertions.

LEMMA 4.2.1. *Let $\xi \geq 0$, for $\lambda \geq 0$ let*

$$\widetilde{F}(\lambda) = \exp\left(-\int_0^\infty \left(1 - e^{-\lambda t}\right) G(dt)\right). \tag{4.2.10}$$

If the function

$$L(t) = \int_1^t G(dy)$$

is continuous on the set $(0, b]$, so is $F(t)$.

LEMMA 4.2.2. *Let the measure G in* (4.2.10) *be absolutely continuous, and let its density $g(t)$ be continuous for $t > \alpha$, be right-continuous at the point $t = \alpha$, and $g(t) = 0$ for $0 \le t < \alpha$. Then $F(t)$ is differentiable for $t > \alpha$, and the derivative $F'(t) = f(t)$ for $t > \alpha$ obeys the relation*

$$tf(t) = \int_0^t (t-y)g(t-y)\,dF(y). \tag{4.2.11}$$

LEMMA 4.2.3. *Let the measure G in* (4.2.10) *be absolutely continuous, and let its density $g(t)$ be continuous on* $[0,\infty)$. *Then $F(t)$ is differentiable for $t > 0$, and its derivative $F'(t) = f(t)$ obeys* (4.2.11) *for $t > 0$.*

LEMMA 4.2.4. *Let $\xi \ge 0$ and relations* (4.2.4), (4.2.5), (4.2.10) *hold. If $g(t) = 0$ for $t > \alpha$, $\alpha > 0$, then the density $f(t)$ of the distribution of ξ is*

$$f(t) = o(t^{-n}), \qquad t \to \infty, \tag{4.2.12}$$

for any fixed $n \in \mathbf{N}$.

We first derive Theorems 4.2.1 and 4.2.2 from Lemmas 4.2.1–4.2.4, and then prove the lemmas.

PROOF OF THEOREM 4.2.1. From Lemma 4.2.3 and relation (4.2.10) it follows that for $\lambda \ge 0$

$$\widetilde{F}(\lambda) = \exp(-r(0)) + \widehat{f}(\lambda) = \exp\left(-\int_0^\infty (1-e^{-\lambda t})g(t)\,dt\right). \tag{4.2.13}$$

Differentiating (4.2.13) n times as a composite function, we arrive at

$$\widehat{f}^{(n)}(\lambda) = \widetilde{F}(\lambda) \sum_{i_1,\dots,i_n} C(i_1,\dots,i_n) \prod_{j=1}^n (\widehat{g}^{(j)}(\lambda))^{i_j}, \tag{4.2.14}$$

where the summation is over all ordered tuples of non-negative integers $i_1,\dots,i_n$ which obey the condition

$$\sum_{j=1}^n j i_j = n,$$

and $C(i_1,\dots,i_n)$ are some constants. Let $k \in \mathbf{N}$ be chosen so that

$$I_k = \int_0^\infty t^k g(t)\,dt = \infty.$$

We fix an arbitrary $n \ge k$. Then, by virtue of Lemma 4.1.7, for all $i_1,\dots,i_n$ such that $i_n = 0$ we obtain

$$\prod_{j=1}^n (\widehat{g}^{(j)}(\lambda))^{i_j} = o(|\widehat{g}^{(n)}(\lambda)|), \qquad \lambda \downarrow 0. \tag{4.2.15}$$

Since $C(0, 0, \dots, 0, 1) = 1$, from (4.2.14), taking (4.2.15) into account, we obtain

$$\widehat{f}^{(n)}(\lambda) = (1 + o(1))\widehat{g}^{(n)}(\lambda), \qquad \lambda \downarrow 0. \tag{4.2.16}$$

We set

$$a(t) = tf(t), \quad b(t) = tg(t), \quad t \geq 0.$$

From (4.2.16) it follows that relation (1.6.24) holds for all $n \geq k - 1$. Furthermore, (1.6.22) follows from (4.2.1). In order to make use of Theorem 1.6.5, it remains to show that (1.6.23) is true. By virtue of Lemma 4.2.3, equality (4.2.11) is true for $t > 0$. In terms of the functions $a(t)$ and $b(t)$, this equality takes the form

$$a(t) = \int_0^t b(t - u)\, dF(u). \tag{4.2.17}$$

From (4.2.17) it follows that

$$\begin{aligned} a(y) - a(x) &= \int_0^x (b(y - u) - b(x - u))\, dF(u) + \int_x^y b(y - u)\, dF(u) \\ &\leq \int_{x-\alpha}^x (b(y - u) - b(x - u))\, dF(u) + \int_x^y b(y - u)\, dF(u), \end{aligned}$$

for $x \geq \alpha$ and $y \geq x$, because by the hypothesis of the theorem the function $b(t)$ does not increase for $t \geq \alpha$. Therefore,

$$a(y) - a(x) \leq \int_{x-\alpha}^y b(y - u)\, dF(u) \leq \sup_{v \geq 0} b(v)(T(x - \alpha) - T(y)), \tag{4.2.18}$$

where

$$T(u) = 1 - F(u), \qquad u \geq 0.$$

We observe that

$$\begin{aligned} 0 \leq \frac{r(x) - r(y)}{r(x)} &= \frac{\int_x^y g(u)\, du}{\int_x^\infty g(u)\, du} \leq \frac{\int_x^y g(u)\, du}{\int_x^{2x} g(u)\, du} \\ &\leq \frac{g(x)(y - x)}{xg(2x)} = \left(\frac{y}{x} - 1\right) \frac{g(x)}{g(2x)} \end{aligned}$$

for $y \geq x \geq \alpha$ because of the monotonicity of $g(x)$. The last expression is $o(1)O(1) = o(1)$ as $x \to \infty$, $y = x + o(x)$ according to (4.2.1). Therefore, $r(x)$ is weakly oscillating at infinity. So, with the use of Corollary 4.1.1, hence we obtain

$$T(x) = (1 + o(1))r(x), \qquad x \to \infty. \tag{4.2.19}$$

We observe that condition (4.2.2) is equivalent to

$$\liminf_{t\to\infty} \frac{\int_t^{t\lambda} g(x)\, dx}{\int_t^\infty g(x)\, dx} > 0. \tag{4.2.20}$$

In its turn, (4.2.20) is equivalent to the inequality

$$\limsup_{t\to\infty} \frac{r(t)}{\int_t^{t\lambda} g(x)\,dx} < \infty. \tag{4.2.21}$$

But relation (4.2.21), because $g(x)$ is dominatedly varying, holds true if and only if

$$\limsup_{t\to\infty} \frac{r(t)}{tg(t)} < \infty. \tag{4.2.22}$$

From (4.2.19) it follows that

$$\begin{aligned} T(x-\alpha) - T(y) &= (1+o(1))r(x-\alpha) - (1+o(1))r(y) \\ &= r(x-\alpha) - r(y) + o(r(x)) = o(r(x)) \end{aligned}$$

for $y \geq x$ as $x \to \infty$, $y = x + o(x)$ because, as we have seen, $r(x)$ is weakly oscillating. In view of (4.2.22), hence we obtain

$$T(x-\alpha) - T(y) = o(xg(x)) \tag{4.2.23}$$

for $y \geq x$ as $x \to \infty$, $y = x + o(x)$. Relation (1.6.23) now follows from (4.2.18) and (4.2.23). Theorem 1.6.5 now yields $a(t) \overset{w}{\sim} b(t)$ as $t \to \infty$, which, in turn, yields $f(t) \overset{w}{\sim} g(t)$. Thus, the theorem is proved in the special case under consideration. By the way, $f(x)$ is bounded for $x > 0$: by Lemma 4.2.3, for $x > 0$

$$\begin{aligned} f(x) &= \frac{1}{x}\int_0^x (x-u)g(x-u)\,dF(u) \leq \int_0^x g(x-u)\,dF(u) \\ &\leq \sup_{v>0} g(v)F(x) \leq \sup_{v>0} g(v). \end{aligned}$$

In other words, under the hypotheses of the theorem

$$\sup_{x>0} f(x) \leq \sup_{x>0} g(x) < \infty.$$

In the general case, the distribution of the random variable ξ admits the representation (Feller, 1966, Section XVII.4 (d)) as the sum $\xi_1 + \xi_2 + \xi_3$ of three independent infinitely divisible random variables ξ_1, ξ_2, ξ_3 such that $\xi_1 \geq 0$ has the spectral measure $\chi\{(0,\infty)\}(x)G(dx)$, $\xi_3 \leq 0$, and the distribution of ξ_3 has all moments. We set

$$\begin{aligned} H(x) &= \mathsf{P}\{\xi_2 + \xi_3 \leq x\}, \qquad x \in \mathbf{R}^1, \\ p(x) &= \frac{d}{dt}\mathsf{P}\{\xi_1 \leq x\}, \qquad x \neq 0. \end{aligned}$$

We observe that for any $n \in \mathbf{N}$, as $t \to \infty$,

$$1 - H(t) = \mathsf{P}\{\xi_1 + \xi_2 > t\} \leq \mathsf{P}\{\xi_3 > t\} = o(t^{-n}). \tag{4.2.24}$$

We represent

$$f(t) = \int_{-\infty}^{\infty} p(t-u)\,dH(u)$$

as the sum of three integrals

$$\begin{aligned} f(t) &= \int_{-\sqrt{t}}^{\sqrt{t}} p(t-u)\,dH(u) + \int_{\sqrt{t}}^{\infty} p(t-u)\,dH(u) + \int_{-\infty}^{-\sqrt{t}} p(t-u)\,dH(u) \\ &= I_1(t) + I_2(t) + I_3(t). \end{aligned} \tag{4.2.25}$$

According to the abovesaid,

$$p(t) \overset{w}{\sim} g(t), \qquad t \to \infty. \tag{4.2.26}$$

We fix arbitrary $\varepsilon, \delta \in (0, 1)$ and set

$$\delta_1 = 1 - (1-\delta)^2.$$

By (4.2.26), there exists $x_0 > 0$ such that for $x \geq x_0$

$$p(x) \leq (1+\varepsilon) g(x(1-\delta_1)). \tag{4.2.27}$$

Let t_0 solve the equation

$$t - \sqrt{t} = \max(x_0, \alpha/(1-\delta_1)).$$

Then for $t \geq t_0$, in view of (4.2.27),

$$\begin{aligned} I_1(t) &= \int_{-\sqrt{t}}^{\sqrt{t}} p(t-u)\,dH(u) \leq \sup_{|t-x|\leq\sqrt{t}} p(x) \\ &\leq (1+\varepsilon) \sup_{|t-x|\leq\sqrt{t}} g(x(1-\delta_1)) \leq (1+\varepsilon) g((t-\sqrt{t})(1-\delta_1)), \end{aligned} \tag{4.2.28}$$

because by the hypothesis of the theorem $g(u)$ does not increase for $u \geq \alpha$. We observe also that for $t \geq \delta_1^{-2}$ the inequality $t - \sqrt{t} \geq t(1-\delta_1)$ is true. Therefore, from (4.2.28) it follows that

$$I_1(t) \leq (1+\varepsilon) g(t(1-\delta_1)^2) = (1+\varepsilon) g(t(1-\delta)) \tag{4.2.29}$$

for $t \geq t_1 = \max(t_0, \delta_1^{-2}, \alpha(1-\delta_1)^{-2})$ by the definition of δ_1. Next, we set

$$\delta_2 = \sqrt{1+\delta} - 1.$$

By (4.2.26), there exists $x_1 > 0$ such that for $x \geq x_1$

$$p(x) \geq (1-\varepsilon) g(x(1+\delta_2)). \tag{4.2.30}$$

Let t_2 solve the equation

$$t - \sqrt{t} = \max(x_1, \alpha/(1+\delta_2)).$$

Then for $t \geq t_2$, in view of (4.2.30),

$$I_1(t) = \int_{-\sqrt{t}}^{\sqrt{t}} p(t-u)\,dH(u) \geq (H(\sqrt{t}) - H(-\sqrt{t})) \inf_{|t-x|\leq\sqrt{t}} p(x)$$
$$\geq (1-\varepsilon)g((t+\sqrt{t})(1+\delta_2))(H(\sqrt{t}) - H(-\sqrt{t})) \tag{4.2.31}$$

because of the monotonicity of $g(x)$. We observe also that for $t \geq \delta_2^{-2}$ the inequality $t + \sqrt{t} \geq t(1+\delta_2)$ is true. Therefore, from (4.2.31) it follows that

$$I_1(t) \geq (1-\varepsilon)g(t(1+\delta_2)^2)(H(\sqrt{t}) - H(-\sqrt{t}))$$
$$= (1-\varepsilon)g(t(1+\delta))(H(\sqrt{t}) - H(-\sqrt{t})) \tag{4.2.32}$$

for $t \geq t_3 = \max(t_2, \delta_2^{-2})$ by the definition of δ_2. Since

$$H(\sqrt{t}) - H(-\sqrt{t}) \to 1, \qquad t \to \infty,$$

from (4.2.29) and (4.2.32) it follows that

$$I_1(t) \overset{w}{\sim} g(t), \qquad t \to \infty. \tag{4.2.33}$$

By (4.2.24), as $t \to \infty$

$$I_2(t) = \int_{\sqrt{t}}^{\infty} p(t-u)\,dH(u) \leq \sup_{x>0} p(x)(1 - H(\sqrt{t})) = o(g(t)), \tag{4.2.34}$$

because the dominatedly varying function $g(x)$ is bounded below by some exponential function (Seneta, 1976, Theorem A.5). There exists a constant C such that for $t \geq \alpha$

$$I_3(t) = \int_{-\infty}^{-\sqrt{t}} p(t-u)\,dH(u) \leq C \int_{-\infty}^{-\sqrt{t}} g(t-u)\,dH(u)$$
$$\leq Cg(t)H(-\sqrt{t}) = o(g(t)), \qquad t \to \infty. \tag{4.2.35}$$

Now (4.2.3) follows from relations (4.2.33)–(4.2.35) and the fact that

$$g((1-\delta)t) \asymp g(t) \asymp g((1+\delta)t), \qquad t \to \infty,$$

for any fixed $\delta \in (0,1)$. The theorem is thus proved. □

PROOF OF THEOREM 4.2.2. In the case where

$$G(dx) = g(x)\chi\{[\alpha,\infty)\}(x)\,dx,$$

the reasoning is the same as above, but we use Lemma 4.2.2 instead of Lemma 4.2.3. Then we assume that ξ is a non-negative random variable which satisfies (4.2.10). Let $P(x)$ be the distribution function of the infinitely divisible random variable with spectral measure $g(x)\chi\{[\alpha,\infty)\}(x)\,dx$, $p(t)$ be its derivative for $t \geq \alpha$ (see Lemma 4.2.2), let $H(t)$ be the distribution function of the infinitely divisible random variable with spectral measure $g(x)\chi\{[0,\alpha)\}(x)\,dx + G_B(dx)$, $h(t)$ be its density. Then for $t > \alpha$

$$f(t) = \int_0^t h(u)\,dP(t-u) = \int_0^{t-\alpha} h(u)p(t-u)\,du + h(t)P(0)$$
$$= \int_0^{t-\alpha} p(t-u)\,dH(u) + h(t)P(0).$$

We represent $f(t)$ as the sum of three terms

$$f(t) = \int_0^{\sqrt{t}} p(t-u)\,dH(u) + \int_{\sqrt{t}}^{t-\alpha} p(t-u)\,dH(u) + h(t)P(0)$$
$$= I_1(t) + I_2(t) + I_3(t).$$

In the same way as in the course of proof of Theorem 4.2.1 we demonstrate that

$$I_1(t) \overset{w}{\sim} p(t), \qquad t \to \infty.$$

For $I_2(t)$, the estimate

$$I_2(t) = \int_{\sqrt{t}}^{t-\alpha} p(t-u)\,dH(u) \le \sup_{x>\alpha} p(x)(1 - H(\sqrt{t})) = o(g(t))$$

holds true as $t \to \infty$. By virtue of Lemma 4.2.4,

$$I_3(t) = h(t)P(0) = o(g(t)), \qquad t \to \infty.$$

The asymptotic relations obtained for I_1, I_2, I_3 yield

$$f(t) \overset{w}{\sim} g(t), \qquad t \to \infty.$$

The remaining reasoning repeats that used in the proof of Theorem 4.2.1. □

Let us turn to proving Lemmas 4.2.1–4.2.4. We set

$$F_1(t) = \int_0^t u\,dF(u), \quad G_1(t) = \int_0^t u\,G(du), \quad t \ge 0.$$

Differentiating (4.2.10) with respect to λ, we obtain

$$\widetilde{F}_1(\lambda) = \widetilde{F}(\lambda)\widetilde{G}_1(\lambda),$$

hence for $t \ge 0$

$$F_1(t) = \int_0^t G_1(t-u)\,dF(u). \tag{4.2.36}$$

PROOF OF LEMMA 4.2.1. From the equality

$$G_1(t+h) - G_1(t) = \int_t^{t+h} u\, G(du) = (t + \theta h)(L(t+h) - L(t)), \tag{4.2.37}$$

which is true for all $t, h \geq 0$, where $\theta \in [0,1]$, it follows that the function $G_1(t)$ is continuous on $[0, b]$. We fix an arbitrary $t \in (0, b)$. Then for $h > 0$ according to (4.2.36)

$$F_1(t+h) - F_1(t) = \int_t^{t+h} G_1(t+h-u)\, dF(u) + \int_0^t (G_1(t+h-u) - G_1(t-u))\, dF(u) = I_1 + I_2.$$

For I_1 the inequalities

$$0 \leq I_1 \leq G_1(h)F(t+h) \leq G_1(h) = o(1), \qquad h \downarrow 0,$$

are true. Due to the uniform continuity of $G_1(t)$ on $[0, b]$, we find that $I_2 = o(1)$ as $h \downarrow 0$, so $F_1(t)$ is continuous on $(0, b)$. But for $t, h > 0$, there exists $\theta \in [0, 1]$ such that

$$F_1(t+h) - F_1(t) = \int_t^{t+h} u\, dF(u) = (t + \theta h)(F(t+h) - F(t)),$$

which implies continuity of $F(t)$ on the set $(0, b)$. The lemma is proved. □

PROOF OF LEMMA 4.2.2. Let $t > 0$. Then there is $\theta \in [0, 1]$ such that

$$\frac{1}{h}\left(\int_0^{t+h} u\, dF(u) - \int_0^t u\, dF(u)\right) = (t + \theta h)\frac{F(t+h) - F(t)}{h}.$$

Then $F(t)$ has the derivative at $t > 0$ if and only if there exists the derivative of $F_1(t)$ at t. Moreover,

$$F_1'(t) = tF'(t) = tf(t).$$

Let us check whether there exists the derivative of $F_1(t)$ or not. Let $0 < h < \alpha < t$. Then

$$\int_0^{t+h} G_1(t+h-u)\, dF(u) - \int_0^t G_1(t-u)\, dF(u) = \int_t^{t+h} G_1(t+h-u)\, dF(u) + \int_0^t (G_1(t+h-u) - G_1(t-u))\, dF(u).$$

But

$$\int_t^{t+h} G_1(t+h-u)\, dF(u) = 0.$$

Therefore, by virtue of (4.2.30)

$$\begin{aligned} F_1(t+h) - F_1(t) &= \int_0^t (G_1(t+h-u) - G_1(t-u))\, dF(u) \\ &= \int_{t-\alpha}^{t-\alpha+h} G_1(t+h-u)\, dF(u) \\ &\quad + \int_0^{t-\alpha} (G_1(t+h-u) - G_1(t-u))\, dF(u). \end{aligned}$$

Let I_1 denote the former integral and I_2, the latter. First, there exist $\theta_1, \theta_2 \in [0, 1]$ such that for $b(t) = tg(t)$

$$I_1 = \int_{t-\alpha}^{t-\alpha+h} G_1(t+h-u)\,dF(u) = G_1(\alpha+\theta_1 h)(F(t-\alpha+h) - F(t-\alpha))$$
$$= \theta_1 h b(\alpha+\theta_2 h)(F(t-\alpha+h) - F(t-\alpha)) = ho(1), \qquad h \downarrow 0,$$

because F is continuous at the point $t-\alpha$ by virtue of Lemma 4.2.1. Second,

$$I_2 = \int_0^{t-\alpha} (G_1(t+h-u) - G_1(t-u))\,dF(u) = \int_0^{t-\alpha} \left(\int_{t-u}^{t+h-u} b(v)\,dv \right) dF(u)$$
$$= h\int_0^{t-\alpha} b(t-u+\theta_3 h)\,dF(u) = h\left(\int_0^{t-\alpha} b(t-u)\,dF(u) + o(1) \right), \qquad h \downarrow 0,$$

for some $\theta_3 \in [0, 1]$ because of the uniform continuity of $b(x)$ on the set $[\alpha, t]$. The case $h < 0$ is treated similarly. Thus, at the point t the function F_1 has the derivative $f_1(t)$, and

$$f_1(t) = tf(t) = \int_0^t b(t-u)\,dF(u).$$

The lemma is thus proved. □

Lemma 4.2.3 is validated with the use of the same reasoning as Lemma 4.2.2, so we omit the proof.

PROOF OF LEMMA 4.2.4. First we assume that $G = G_A$ and

$$g(t) = \begin{cases} 1/t, & t \in (0, \varepsilon), \\ 0, & t > \varepsilon. \end{cases}$$

Then by (4.2.36) for $t > 0$

$$F_1(t) = \int_0^t F(t-u)\,dG_1(u) = \int_0^t F(t-u)\,du, \qquad t \le \varepsilon,$$

and

$$F_1(t) = \int_{t-\varepsilon}^t F(u)\,du, \qquad t > \varepsilon.$$

Hence it follows that $F(t)$ is differentiable for $t \ge 0$, whereas $f(t) = F'(t)$ obeys the equalities

$$tf(t) = F(t), \qquad t \le \varepsilon \tag{4.2.38}$$

and

$$tf(t) = F(t) - F(t-\varepsilon), \qquad t > \varepsilon. \tag{4.2.39}$$

Since the distribution function $F(t)$ has all moments, we see that for any $n \in \mathbf{N}$, as $t \to \infty$,

$$f(t) = t^{-1}(F(t) - F(t-\varepsilon) \le t^{-1}(1 - F(t-\varepsilon)) = o(t^{-n}). \tag{4.2.40}$$

In the general case, $f(t)$ admits the representation as the convolution

$$f(t) = \int_0^t p(t-u)\,dH(u),$$

where $p(t)$ is the density of the infinitely divisible law with spectral measure $t^{-1} \times \chi\{(0,\varepsilon)\}(t)\,dt$, $H(t)$ is the distribution function of the infinitely divisible law with spectral measure $(g(t) - t^{-1}\chi\{(0,\varepsilon)\}(t))\,dt$. So we obtain

$$f(t) = \int_0^{t/2} p(t-u)\,dH(u) + \int_{t/2}^t p(t-u)\,dH(u) = I_1 + I_2. \tag{4.2.41}$$

By virtue of (4.2.40), for $f = p$

$$I_1 \le H(t/2) \sup_{y \ge t/2} p(y) = o(t^{-n}), \qquad t \to \infty. \tag{4.2.42}$$

It is not difficult to see that

$$c = \sup_{t \ge 0} p(t) < \infty. \tag{4.2.43}$$

From (4.2.38) with $f = p$ it follows indeed that $p(t) = c_0$ for $t \in [0, \varepsilon]$, where c_0 is some constant depending on ε. From (4.2.35) we see that

$$\sup_{t > \varepsilon} p(t) < \infty.$$

Therefore, inequality (4.2.43) implies the inequality

$$I_2 \le c(H(t/2) - H(t)) \le c(1 - H(t/2)) = o(t^{-n}), \qquad t \to \infty, \tag{4.2.44}$$

because the distribution function $H(t)$ has all moments. Now (4.2.12) follows from (4.2.41), (4.2.42), and (4.2.44). The lemma is proved. □

REMARK 4.2.1. From inequality (4.2.43) it follows that condition (4.2.5) is sufficient not only for the density of an infinitely divisible distribution to exist, but also for it to be bounded.

4.3. Multidimensional case

Let Γ be an arbitrary closed convex acute solid homogeneous cone in $\mathbf{R}^n$ with apex at zero (see the beginning of Section 1.1 and Definition 1.8.6). For $x \in \Gamma$, we set

$$\Gamma(x) = \{y : y \in \Gamma,\ x - y \in \Gamma\} = \{y : y \overset{\Gamma}{\le} x\}, \qquad \widetilde{\Gamma}(x) = \Gamma \setminus \Gamma(x). \tag{4.3.1}$$

We begin with the following lemma (see (Yakymiv, 1997, Lemma 3)).

LEMMA 4.3.1. *Let ν be a σ-finite measure on Γ (maybe unbounded in some neighbourhood of zero) which obeys the relations*

$$\int_{|x|\le 1} |x|\nu(dx) < \infty, \qquad \int_{|x|>1} \nu(dx) < \infty.$$

Then there exists a random vector $\xi \in \mathbf{R}^n$ taking values in Γ, and its Laplace transform is of the form

$$\mathsf{E}e^{-(\lambda,\xi)} = \exp\left(-\int_\Gamma \left(1 - e^{-(\lambda,x)}\right)\nu(dx)\right), \qquad \lambda \in \Gamma^*. \tag{4.3.2}$$

A measure ν is said to be the Lévy spectral measure of the distribution of the random vector ξ. For $x \in G$ we set

$$f(x) = \mathsf{P}\{\xi \in \widetilde{\Gamma}(x)\}, \qquad g(x) = \nu(\widetilde{\Gamma}(x)). \tag{4.3.3}$$

The following limit theorem is true.

THEOREM 4.3.1. *Let a function $g(x)$ be an admissible function of first type for the cone Γ ($g(x) \in D_1(\Gamma)$, see Definition 1.8.5). Then*

$$f(x) = (1 + o(1))g(x), \quad x \in G, \quad \Delta_\Gamma(x) \to \infty, \tag{4.3.4}$$

that is, for $x \in G = \operatorname{int}\Gamma$, as $\Delta_\Gamma(x) \to \infty$,

$$\mathsf{P}\{\xi \in \widetilde{\Gamma}(x)\} = (1 + o(1))\nu(\widetilde{\Gamma}(x)),$$

where $\Delta_\Gamma(x)$ is the distance between the point x and the boundary of the cone Γ.

This theorem is proved in (Yakymiv, 2003b).

REMARK 4.3.1. Theorem 4.3.1 generalises the corresponding assertions in (Omey, 1985a; Yakymiv, 1997), where $\Delta_\Gamma(x)$ was of order of magnitude $|x|$ as $|x| \to \infty$, whereas in our presentation $\Delta_\Gamma(x)$ tends to infinity as slow as we wish.

We prove Theorem 4.3.1 as a corollary to assertion 1 of Theorem 1.8.3. It would be alluring to prove, with the use of assertion 2 (respectively 3) of that theorem, that (4.3.4) holds if $|x| \to \infty$ and $\Delta_\Gamma(x) \ge \delta > 0$ (respectively, if $|x| \to \infty$ and $x \in G$). But the following elementary arguments show that (4.3.4), generally speaking, does not hold even if $|x| \to \infty$ and $\Delta_\Gamma(x) \ge \delta > 0$. It is easily seen, indeed, that if

$$\Gamma = \mathbf{R}^n_+ = \{x = (x_1, \dots, x_n),\ x_i \ge 0\ \forall i = 1, \dots, n\}$$

and $|x| \to \infty$ so that $x_1 = \delta > 0$, $\min(x_2, \dots, x_n) \to \infty$, then

$$g(x) \to \nu(A),$$

where

$$A = \{y\colon y = (y_1, \dots, y_n) \in \mathbf{R}^n_+,\ y_1 > \delta\},$$

and if $\nu(A) > 1$, then (4.3.4) is obviously broken because

$$f(x) = \mathbf{P}\{\xi \in \widetilde{\Gamma}(x)\} \leq 1.$$

As appropriate examples for $g(x)$, Examples 1.8.1–1.8.3 are good, provided that $g(x)$ is assumed to be non-increasing in Γ.

As above, let $\widehat{f}(y)$ denote the Laplace transform of a function f on Γ:

$$\widehat{f}(y) = \int_\Gamma e^{-(y,x)} f(x)\,dx$$

provided that it exists for $y \in C = \operatorname{int}\Gamma^*$. We set

$$\theta_\Gamma(x) = \begin{cases} 1, & x \in \Gamma, \\ 0, & x \notin \Gamma, \end{cases}$$
$$K_C(\lambda) = \widehat{\theta}_\Gamma(\lambda) = \int_\Gamma e^{-(\lambda,x)}\,dx, \qquad \lambda \in C,$$
$$\nu(x) = \nu\{y\colon y \overset{\Gamma}{\leq} x\}.$$

We prove Theorem 4.3.1 with the use of the following lemma.

LEMMA 4.3.2. *For any $\lambda \in C$, the relation*

$$K_C(\lambda)\int_\Gamma (1 - e^{-(\lambda,x)})\,\nu(dx) = \int_\Gamma e^{-(\lambda,x)} g(x)\,dx < \infty \tag{4.3.5}$$

is true.

PROOF OF LEMMA 4.3.2. First we assume that $\nu(\Gamma) < \infty$. For $\lambda \in C$ we see that

$$\begin{aligned}\int_\Gamma e^{-(\lambda,x)}\nu(x)\,dx &= \int_\Gamma e^{-(\lambda,x)}(\nu * \theta_\Gamma)(x)\,dx \\ &= \int_\Gamma e^{-(\lambda,x)}\nu(dx)K_C(\lambda).\end{aligned}$$

But $g(x) = \nu(\Gamma) - \nu(x)$, therefore

$$\int_\Gamma e^{-(\lambda,x)}\nu(x)\,dx = \nu(\Gamma)K_C(\lambda) - \widehat{g}(\lambda).$$

Thus,

$$\begin{aligned}\widehat{g}(\lambda) &= \nu(\Gamma)K_C(\lambda) - \int_\Gamma e^{-(\lambda,x)}\,\nu(dx)K_C(\lambda) \\ &= \int_\Gamma (1 - e^{-(\lambda,x)})\nu(dx)K_C(\lambda), \qquad \forall\lambda \in C,\end{aligned}$$

which yields (4.3.5) in the case where $\nu(\Gamma) < \infty$.

In the general case, let

$$\Gamma_k = \{u : u \in \Gamma,\ |u| \ge 1/k\}, \quad \nu_k(A) = \nu(A \cap \Gamma_k), \quad g_k(x) = \nu_k(\widetilde{\Gamma}(x))$$

for any $A \in \mathfrak{A}$, $x \in \Gamma$, $k \in \mathbf{N}$, where $\mathfrak{A}$ stands for the set of all bounded Borel sets in $\mathbf{R}^n$. Since the lemma has been proved for the measures ν_k, we see that for any $\lambda \in C$

$$\widehat{g}_k(\lambda) = K_C(\lambda) \int_{\Gamma_k} (1 - e^{-(\lambda,x)})\nu(dx).$$

Therefore, for all $\lambda \in C$, as $k \to \infty$,

$$\widehat{g}_k(\lambda) \to K_C(\lambda) \int_{\Gamma} (1 - e^{-(\lambda,x)})\nu(dx), \tag{4.3.6}$$

because

$$\int_{\Gamma} (1 - e^{-(\lambda,x)})\nu(dx) < \infty$$

for $\lambda \in C$. Since

$$e^{-(\lambda,x)} g_k(x) \uparrow e^{-(\lambda,x)} g(x)$$

for any $x \in G$, $\lambda \in C$, as $k \to \infty$, by virtue of B. Lévy's theorem on monotone convergence, we obtain as $k \to \infty$

$$\widehat{g}_k(\lambda) = \int_{\Gamma} e^{-(\lambda,x)} g_k(x)\, dx \to \int_{\Gamma} e^{-(\lambda,x)} g(x)\, dx < \infty \tag{4.3.7}$$

for all $\lambda \in C$. Comparing (4.3.6) and (4.3.7), we arrive at (4.3.5). The lemma is thus proved. □

PROOF OF THEOREM 4.3.1. From (4.3.2) and (4.3.5) it follows that

$$\mathbf{E} e^{-(\lambda,\xi)} = \exp(-\widehat{g}(\lambda)/K_C(\lambda)), \qquad \forall \lambda \in C. \tag{4.3.8}$$

Let P be the distribution of the random vector ξ. As in the course of proof of (4.3.5), we find that

$$1 - \widetilde{P}(\lambda) = \int_{\Gamma} (1 - e^{-(\lambda,x)}) P(dx) = \widehat{f}(\lambda)/K_C(\lambda), \qquad \forall \lambda \in C. \tag{4.3.9}$$

From (4.3.8) and (4.3.9) it follows that

$$\frac{\widehat{f}(\lambda)}{K_C(\lambda)} = 1 - \exp\left(-\frac{\widehat{g}(\lambda)}{K_C(\lambda)}\right), \qquad \forall \lambda \in C. \tag{4.3.10}$$

For $\lambda \in C$ we set $\zeta_\lambda = (\lambda, \xi)$. We obtain

$$\zeta_\lambda \ge 0, \qquad \forall \lambda \in C,$$

$$\zeta_\lambda \xrightarrow{P} 0, \quad \lambda \to 0, \quad \lambda \in C.$$

Since the function $\exp(-t)$ for $t \geq 0$ is continuous and bounded, we see that

$$\mathbf{E}e^{-(\lambda,\xi)} = \mathbf{E}e^{-\zeta_\lambda} \to 1, \quad \lambda \to 0, \quad \lambda \in C. \tag{4.3.11}$$

From (4.3.8) and (4.3.11) it follows that

$$\frac{\widehat{g}(\lambda)}{K_C(\lambda)} = -\ln \mathbf{E}e^{-(\lambda,\xi)} \to 0, \quad \lambda \to 0, \quad \lambda \in C. \tag{4.3.12}$$

From (4.3.10) and (4.3.12) we arrive at

$$\frac{\widehat{f}(\lambda)}{K_C(\lambda)} = (1+o(1))\frac{\widehat{g}(\lambda)}{K_C(\lambda)}, \quad \lambda \to 0, \quad \lambda \in C,$$

or, what is the same,

$$\widehat{f}(\lambda) = (1+o(1))\widehat{g}(\lambda), \quad \lambda \to 0, \quad \lambda \in C.$$

We see that the hypotheses of assertion 1 of Theorem 1.8.3 are satisfied, which states that relation (4.3.4) is true. The theorem is proved. □

To close the section, we prove Lemma 4.3.1.

PROOF OF LEMMA 4.3.1. We set

$$\varphi(\lambda) = \exp\left(-\int_\Gamma (1 - e^{-(\lambda,x)})\nu(dx)\right), \qquad \lambda \in \Gamma^*.$$

It is clear that it is sufficient to assume that $\nu(\Gamma \setminus \{0\}) > 0$. Then $\nu(\Gamma_m) > 0$ for some $m \in \mathbf{N}$, where

$$\Gamma_k = \{x : x \in \Gamma, \|x| \geq 1/k\}, \qquad k \in \mathbf{N}.$$

For $k \in \mathbf{N}$, $k \geq m$ we set

$$H_k(A) = \frac{\nu(A \cap \Gamma_k)}{\nu(\Gamma_k)}, \qquad A \in \mathfrak{A}.$$

It is clear that H_k is a probability distribution in $\mathbf{R}^n$ concentrated in Γ_k. We set

$$\theta_k = 1/\nu(\Gamma_k), \quad k \in \mathbf{N}, \quad k \geq m.$$

Let $X_1, X_2, \dots$ be independent random vectors in $\mathbf{R}^n$ with distribution H_k, and let Y be an independent of them Poisson random variable with parameter θ_k. Then

$$\begin{aligned}\mathbf{E}\exp\left(-\left(\lambda, \sum_{j=0}^{Y} X_j\right)\right) &= \exp(-\theta_k(1 - \mathbf{E}e^{-(\lambda,X_1)})) \\ &= \exp\left(-\int_{\Gamma_k} (1 - e^{-(\lambda,x)})\nu(dx)\right) \equiv \varphi_k(\lambda)\end{aligned}$$

is the Laplace transform of some infinitely divisible measure P_k on Γ for $\lambda \in \Gamma^*$. Since $\varphi_k(\lambda) \to \varphi(\lambda)$ as $k \to \infty$, we conclude that $\varphi(\lambda)$ is the Laplace transform of some infinitely divisible measure P on Γ. The lemma is thus proved. □

5

Limit theorems in the record model

5.1. Intervals between state change times in the record process

Let independent random variables $\xi_1, \xi_2, \xi_3, \ldots, \eta_0, \eta_1, \eta_2, \ldots$ be given such that $\mathbf{P}\{\xi_n \le x\} = F(x)$, $n \in \mathbf{N}$, and $\mathbf{P}\{\eta_n \le x\} = G(x)$, $n \in \mathbf{Z}_+ = \mathbf{N} \cup \{0\}$. We assume that $F(0) = 0$ and $G(x)$ is continuous. We set

$$K = \{n : n \in \mathbf{N},\ \eta_n > \eta_m\ \forall m < n,\ m \in \mathbf{Z}_+\} \cup \{0\}, \quad S_n = \sum_{k=1}^{n} \xi_k, \quad n \in \mathbf{N}, \quad S_0 = 0,$$

$$N(t) = \max\{n : n \in \mathbf{Z}_+,\ S_n \le t\}, \qquad M(t) = \max\{n : n \in K,\ n \le N(t)\},$$

$$T(t) = 1 - F(t), \quad V(t) = \int_0^t T(u)\,du, \quad t \ge 0.$$

A *stochastic record process* is the process $\{\eta_{M(t)},\ t \ge 0\}$. The *record times* are the random variables $\{\nu_n,\ n \in \mathbf{Z}_+\}$, which are the times of the nth jump in the stochastic record process $\{\eta_{M(t)},\ t \ge 0\}$, where $\nu_0 = 0$. In this section we study the asymptotic properties of the tails of distributions of $\tau_n = \nu_n - \nu_{n-1}$, $n \in \mathbf{N}$. The following three limit theorems are true. Recall that the definition of weak equivalence of functions is given in the beginning of Section 1.6.

THEOREM 5.1.1. *For all* $\lambda \in (0,1)$, *let*

$$\limsup_{t \to \infty} V(\lambda t)/V(t) < 1. \tag{5.1.1}$$

Then for any fixed $n \in \mathbf{N}$, *as* $t \to \infty$,

$$\mathbf{P}\{\tau_n > t\} \overset{w}{\sim} T(t) L^n(t)/n!, \tag{5.1.2}$$

where $L(t)$ *is a non-decreasing slowly varying at infinity function equal to* $\ln(t/V(t))$.

THEOREM 5.1.2. *Let* $\mathbf{E}\xi_1 < \infty$ *and* $T(t) = o((t \ln t)^{-1})$ *as* $t \to \infty$. *Then for any fixed* $n \in N$, *as* $t \to \infty$,

$$\mathbf{P}\{\tau_n > t\} \sim \mu t^{-1} \ln^{n-1}(t)/(n-1)!$$

THEOREM 5.1.3. *For all* $\lambda \in (0,1)$ *and some* $n \in \mathbf{N}$, *let*

$$\limsup_{t\to\infty} W_n(\lambda t)/W_n(t) < 1,$$

where

$$W_n(t) = \int_0^t u f_n(u)\,du, \qquad f_n(t) = L^n(t)(T(t) + nt^{-1}V(t)/L(t)).$$

Then, as $t \to \infty$,

$$\mathsf{P}\{\tau_n > t\} \overset{w}{\sim} f_n(t)/n!$$

Theorem 5.1.1 covers the cases where $\mathsf{E}\xi_1 = \infty$ and the function $g(t) = t\mathsf{P}\{\xi > t\}$ is not slowly varying. Theorem 5.1.2 is true for $\mathsf{E}\xi_1 < \infty$ and $g(t) = o(1/\ln t)$, $t \to \infty$. Theorem 5.1.3 is true in the 'intermediate' case where $g(t)$ is a slowly varying function. The same assertions are true for the random variables $\{\nu_n,\ n \in \mathbf{N}\}$; the proofs are also similar.

COROLLARY 5.1.1. *Let* $T(t)$ *be regularly varying at infinity with index* $-\alpha$, *where* $\alpha \in (0,1)$. *Then for any fixed* $n \in \mathbf{N}$, *as* $t \to \infty$,

$$\mathsf{P}\{\tau_n > t\} \sim \alpha^n T(t)\ln^n(t)/n!$$

COROLLARY 5.1.2. *Let* $T(t)$ *be a slowly varying function. Then for any fixed* $n \in \mathbf{N}$, *as* $t \to \infty$,

$$\mathsf{P}\{\tau_n > t\} \sim T(t)\ln^n(1/T(t))/n!$$

It is easily seen, indeed, that if $T(t)$ is regularly varying at infinity with index $-\alpha$, where $\alpha \in (0,1)$, then, as $t \to \infty$,

$$V(t) = \int_0^t T(u)\,du \sim \frac{tT(t)}{1-\alpha},$$

which proves the corollaries.

Theorem 5.1.3 concerns also cases not covered by Theorems 5.1.1 and 5.1.2. If $T(t) \sim b/t$ as $t \to \infty$, then from Theorem 5.1.3 it follows that

$$\mathsf{P}\{\tau_n > t\} \sim b(n+1)t^{-1}\ln^n(t)/n!$$

If $T(t) \sim b/(t\ln t)$ as $t \to \infty$, then

$$\mathsf{P}\{\tau_n > t\} \sim bt^{-1}\ln^{n-1}(t)\ln\ln t/(n-1)!$$

Theorems 5.1.1–5.1.3 are published in (Yakymiv, 1987a), which continued the studies (Gaver, 1976; Westcott, 1977; Westcott, 1979; Embrechts, Omey, 1983). Surveys of the record model can be found in (Nevzorov, 1987; Nagaraja, 1988; Nevzorov, Balakrishnan, 1998). A wide spectrum of questions of the record theory is covered in (Nevzorov, 2000). In the next section we will speak about the so-called kth record times introduced in (Dziubdziela, 1977; Dziubdziela, Kopocinski, 1976).

Let us turn to proving Theorems 5.1.1–5.1.3. As before, let the Laplace and Laplace–Stieltjes transforms of a function f be denoted by

$$\widehat{f}(\lambda) = \int_0^\infty e^{-t\lambda} f(t)\,dt, \qquad \widetilde{f}(\lambda) = \int_0^\infty e^{-t\lambda}\,df(t).$$

In order to prove Theorems 5.1.1–5.1.3, we need the following five lemmas.

LEMMA 5.1.1. *As* $\lambda \downarrow 0$, $\widehat{T}(\lambda) \asymp V(1/\lambda)$.

LEMMA 5.1.2. *The function* $L(t) = \ln(t/V(t))$ *is a non-decreasing slowly varying one.*

LEMMA 5.1.3. *Let* $l(t)$ *be an arbitrary non-decreasing slowly varying function, and let* (5.1.1) *be true. Then, as* $\lambda \downarrow 0$,

$$\widehat{Tl}(\lambda) \sim \widehat{T}(\lambda) l(1/\lambda).$$

LEMMA 5.1.4. *Let a function* $u(x) \geq 0$ *do not increase, a function* $l(x)$ *do not decrease and be slowly varying at infinity. If*

$$\frac{\int_0^\tau v(x)\,dx}{\int_0^t v(x)\,dx} \to 0 \tag{5.1.3}$$

as $t \to \infty$, $\tau/t \to 0$, *where* $v(x) = xu(x)$, *then*

$$\widehat{v}(\lambda) l(1/\lambda) \sim \widehat{vl}(\lambda), \qquad \lambda \downarrow 0. \tag{5.1.4}$$

LEMMA 5.1.5. *As* $\lambda \downarrow 0$,

$$\widehat{V}(\lambda)/L(1/\lambda) \sim \widehat{V/L}(\lambda). \tag{5.1.5}$$

We first deduce Theorems 5.1.1–5.1.3 from Lemmas 5.1.1–5.1.5, and then prove the lemmas.

PROOF OF THEOREM 5.1.1. We set $Q_{1,m} = 1/m$,

$$Q_{r,m} = m^{-1} \sum_{j=1}^{m} Q_{r-1,j}, \quad r > 1, \quad m \in \mathbf{N}. \tag{5.1.6}$$

For $r \in \mathbf{N}$, $s \in (0,1)$, we introduce the functions

$$G_r(s) = \sum_{m=1}^{\infty} Q_{r,m} s^m, \qquad g_r(s) = (1-s)G_r(s)/s.$$

Since

$$\mathbf{E}\exp(-\lambda\tau_n) = g_n(\widetilde{F}(\lambda)), \qquad n \in \mathbf{N}$$

(see (Embrechts, Omey, 1983, p. 340)), we arrive at

$$\begin{aligned} \widehat{a}(\lambda) &= (1 - g_n(\widetilde{F}(\lambda))/\lambda = ((1-\widetilde{F}(\lambda))/\lambda\widetilde{F}(\lambda))G_n(\widetilde{F}(\lambda)) \\ &= (\widehat{T}(\lambda)/\widetilde{F}(\lambda))G_n(1-\lambda\widehat{T}(\lambda)), \end{aligned} \tag{5.1.7}$$

where

$$a(t) = \mathbf{P}\{\tau_n > t\}.$$

From (5.1.6) it follows that

$$G_r(s) = \int_0^s \frac{G_{r-1}(u)}{u(1-u)}\,du, \qquad s \in (0,1), \tag{5.1.8}$$

for any $r > 1$. From (5.1.8) and the equality $G_1(s) = -\ln(1-s)$ it follows that, as $s \uparrow 1$,

$$G_r(s) = (1+o(1))\ln^r(1/(1-s))/r! \tag{5.1.9}$$

for all $r \in \mathbf{N}$. From (5.1.7) and (5.1.9) we conclude that for any fixed $n \in \mathbf{N}$, as $\lambda \downarrow 0$,

$$\widehat{a}(\lambda) \sim \widehat{T}(\lambda)\ln^n(1/\lambda\widehat{T}(\lambda))/n!;$$

hence, with the use of Lemma 5.1.1, we find that, as $\lambda \downarrow 0$,

$$\widehat{a}(\lambda) \sim \widehat{T}(\lambda)L^n(1/\lambda)/n!$$

Therefore, by virtue of Lemma 5.1.2 and Lemma 5.1.3, as $\lambda \downarrow 0$,

$$\widehat{a}(\lambda) \sim \widehat{b}(\lambda),$$

where

$$b(t) = T(t)L^n(t)/n!$$

We set

$$B(t) = \int_0^t b(u)\,du, \qquad t \geq 0.$$

From (5.1.1) and the relations

$$1 - B(\lambda t)/B(t) = \frac{\int_{\lambda t}^t T(u)L^n(u)\,du}{\int_0^t T(u)L^n(u)\,du} \geq (L^n(\lambda t)/L^n(t))(V(t)-V(\lambda t))/V(t)$$

we arrive at (1.6.2) with $\lambda \in (0,1)$. Making use of Theorem 1.6.1, we find that

$$a(t) \overset{w}{\sim} b(t)$$

as $t \to \infty$. The theorem is proved. □

PROOF OF THEOREM 5.1.2. By differentiating (5.1.7) with respect to λ we obtain

$$\widehat{a}'(\lambda) = \frac{\widehat{T}'(\lambda)}{\widetilde{F}(\lambda)}G_n(\widetilde{F}(\lambda)) + \frac{\widehat{T}(\lambda)}{\widetilde{F}(\lambda)}G_n'(\widetilde{F}(\lambda))\widetilde{F}'(\lambda) - \frac{\widehat{T}(\lambda)}{(\widetilde{F}(\lambda))^2}G_n(\widetilde{F}(\lambda))\widetilde{F}'(\lambda). \tag{5.1.10}$$

Since $\mu = \mathbf{E}\xi_1$ is finite, the last equality can be rewritten as follows:

$$\widehat{a}'(\lambda) = \alpha_1\widehat{T}'(\lambda)G_n(\widetilde{F}(\lambda)) - \alpha_2\mu^2 G_n'(\widetilde{F}(\lambda)) + \alpha_3\mu^2 G_n(\widetilde{F}(\lambda)), \tag{5.1.11}$$

where $\alpha_j \to 1$ as $\lambda \downarrow 0$, $j = 1, 2, 3$. From relations (5.1.8) and (5.1.9) it follows that for any fixed $n \in \mathbf{N}$, as $s \uparrow 1$,

$$G_n'(s) \sim \frac{nG_n(s)}{(1-s)\ln(1/(1-s))}. \tag{5.1.12}$$

Since $T(t) = o((t \ln t)^{-1})$ as $t \to \infty$, we see that

$$\widehat{T}'(\lambda) = o((\lambda \ln(1/\lambda))^{-1}), \qquad \lambda \downarrow 0. \tag{5.1.13}$$

By (5.1.11), (5.1.12), and (5.1.13), as $\lambda \downarrow 0$ we obtain

$$\widehat{a}'(\lambda) \sim -\mu^2 n G_n(\widetilde{F}(\lambda))(\lambda\mu \ln(1/\lambda))^{-1},$$

or, in view of relation (5.1.9),

$$\widehat{a}'(\lambda) \sim -\mu\lambda^{-1} \ln^{n-1}(1/\lambda)/(n-1)!, \qquad \lambda \downarrow 0,$$

which proves the theorem. □

PROOF OF THEOREM 5.1.3. In what follows, $\eta_i = \eta_i(\lambda)$, $i = 1, 2, \ldots, 7$, will be some functions of λ such that $\eta_i(\lambda) \to 1$ as $\lambda \downarrow 0$. From formula (5.1.10) and the equality $\widetilde{F}'(\lambda) = -\lambda\widehat{T}'(\lambda) - \widehat{T}(\lambda)$ we obtain

$$\begin{aligned}\widehat{a}'(\lambda) = G_n(\widetilde{F}(\lambda))(\eta_1\widehat{T}'(\lambda) + \eta_2\widehat{T}(\lambda)(\lambda\widehat{T}'(\lambda) + \widehat{T}(\lambda)))\\ - \eta_3 G_n'(\widetilde{F}(\lambda))\widehat{T}(\lambda)(\lambda\widehat{T}'(\lambda) + \widehat{T}(\lambda)).\end{aligned}$$

Since $\lambda\widehat{T}(\lambda) \to 0$ as $\lambda \downarrow 0$, we see that

$$\widehat{a}'(\lambda) = G_n(\widetilde{F}(\lambda))(\eta_4\widehat{T}'(\lambda) + \eta_2\widehat{T}^2(\lambda)) - \eta_3 G_n'(\widetilde{F}(\lambda))\widehat{T}(\lambda)(\lambda\widehat{T}'(\lambda) + \widehat{T}(\lambda)).$$

Therefore, by formula (5.1.12)

$$\widehat{a}'(\lambda) = G_n(\widetilde{F}(\lambda))\left(\eta_4\widehat{T}'(\lambda) + \eta_2\widehat{T}^2(\lambda) - \eta_5\frac{n(\lambda\widehat{T}'(\lambda) + \widehat{T}(\lambda))}{\lambda\ln(1/\lambda\widehat{T}(\lambda))}\right).$$

Hence

$$\widehat{a}'(\lambda) = G_n(\widetilde{F}(\lambda))\left(\eta_6\widehat{T}'(\lambda) + \eta_2\widehat{T}^2(\lambda) - \eta_5\frac{n\widehat{T}(\lambda)}{\lambda\ln(1/\lambda\widehat{T}(\lambda))}\right).$$

we observe that $\widehat{T}^2(\lambda) = o(\widehat{T}(\lambda)/\lambda)$ as $\lambda \downarrow 0$. Therefore,

$$\widehat{a}'(\lambda) = G_n(\widetilde{F}(\lambda))\left(\eta_6\widehat{T}'(\lambda) - \eta_7\frac{n\widehat{T}(\lambda)}{\lambda\ln(1/\lambda\widehat{T}(\lambda))}\right).$$

As $\lambda \downarrow 0$, hence we obtain

$$\widehat{ta}(\lambda) \sim G_n(\widetilde{F}(\lambda))\left(\widehat{tT}(\lambda) + \frac{n\widehat{V}(\lambda)}{\ln(1/\lambda\widehat{T}(\lambda))}\right),$$

or, in view of formula (5.1.9), as $\lambda \downarrow 0$

$$\widehat{ta}(\lambda) \sim \ln^n(1/\lambda\widehat{T}(\lambda)) \left(\widehat{tT}(\lambda) + \frac{n\widehat{V}(\lambda)}{\ln(1/\lambda\widehat{T}(\lambda))} \right) /n!$$

Therefore, by virtue of Lemma 5.1.1, as $\lambda \downarrow 0$

$$\widehat{ta}(\lambda) \sim L^n(1/\lambda) \left(\widehat{tT}(\lambda) + n\widehat{V}(\lambda)/L(1/\lambda) \right) /n!$$

By virtue of Lemma 5.1.5, hence we obtain, as $\lambda \downarrow 0$,

$$\widehat{ta}(\lambda) \sim L^n(1/\lambda) \left(\widehat{tT}(\lambda) + n\widehat{V/L}(\lambda) \right) /n!$$

Therefore, for $u(t) = T(t) + nV(t)/(tL(t))$

$$\widehat{ta}(\lambda) \sim L^n(1/\lambda)\widehat{tu}(\lambda)/n!$$

By virtue of Lemma 5.1.2, the function $u(t)$ does not increase. Applying Lemma 5.1.4 to the last relation, we see that, as $\lambda \downarrow 0$,

$$\widehat{ta}(\lambda) \sim \widehat{L^n tu}(\lambda)/n!$$

By virtue of Theorem 1.6.1, hence it follows that, as $t \to \infty$,

$$ta(t) \overset{w}{\sim} tu(t)L^n(t)/n!,$$

which implies that

$$\mathbf{P}\{\tau_n > t\} \overset{w}{\sim} f_n(t)/n!$$

as $t \to \infty$. The theorem is proved. □

PROOF OF LEMMA 5.1.1. For $\lambda > 0$,

$$e^{-1}V(1/\lambda) \le \lambda \int_{1/\lambda}^{\infty} e^{-\lambda u} V(u)\, du \le \lambda\widehat{V}(\lambda) = \widehat{T}(\lambda).$$

From Lemma 1.6.1 it follows that the function $V(t) = \int_0^t T(u)\,du$ is weakly oscillating at infinity. Therefore, by virtue of Corollary 1.4.4, there exist positive constants c, b, β such that for $x \ge 1$, $t \ge b$ the inequality

$$V(tx) \le cV(t)x^\beta$$

holds. Therefore, for $\lambda \in (0, 1/b)$

$$\widehat{T}(\lambda) = \lambda\widehat{V}(\lambda) = \int_0^\infty e^{-x}V(x/\lambda)\,dx \le V(1/\lambda)\left(\int_0^1 e^{-x}\,dx + c\int_1^\infty e^{-x}x^\beta\,dx \right).$$

The lemma is proved. □

PROOF OF LEMMA 5.1.2. From Lemma 1.6.1 it follows that $V(t)$ is weakly oscillating at infinity, so $\ln(t/V(t))$ is a slowly varying function. The inequality

$$\frac{d}{dt}(V(t)/t) = t^{-2}\left(tT(t) - \int_0^t T(u)\,du\right) \le 0, \qquad t > 0,$$

implies that $L(t)$ is monotone. The lemma is thus proved. □

PROOF OF LEMMA 5.1.3. Lemma 5.1.3 is true if and only if

$$I(\lambda) = \int_0^\infty e^{-y} T(y/\lambda)\left(\frac{l(y/\lambda)}{l(1/\lambda)} - 1\right) dy = o(R(\lambda)) \tag{5.1.14}$$

as $\lambda \downarrow 0$, where

$$R(\lambda) = \int_0^\infty e^{-y} T(y/\lambda)\,dy.$$

We fix δ and M, $0 < \delta < 1 < M < \infty$, and represent the integral $I(\lambda)$ as the sum of the integrals $I_1(\lambda)$, $I_2(\lambda)$, $I_3(\lambda)$ over the intervals $[0,\delta)$, $[\delta, M]$, (M,∞), respectively:

$$I(\lambda) = I_1(\lambda) + I_2(\lambda) + I_3(\lambda). \tag{5.1.15}$$

By virtue of the uniform convergence theorem for slowly varying functions (Seneta, 1976, Theorem 1.1),

$$I_2(\lambda) = o(R(\lambda)), \qquad \lambda \downarrow 0. \tag{5.1.16}$$

Further,

$$\begin{aligned} 0 \ge I_1(\lambda) &= \int_0^\delta e^{-y} T(y/\lambda)\left(\frac{l(y/\lambda)}{l(1/\lambda)} - 1\right) dy \\ &\ge -\int_0^\delta e^{-y} T(y/\lambda)\,dy = -\lambda \int_0^{\delta/\lambda} e^{-\lambda x} T(x)\,dx \ge -\lambda V(\delta/\lambda). \end{aligned} \tag{5.1.17}$$

From the integral representation theorem for slowly varying functions (Seneta, 1976, Theorem 1.2) it follows that $l(x)/l(y) \le c(x/y)^\varepsilon$ for some $\varepsilon, b, c > 0$ and all $x \ge y \ge b$. Therefore, for any $\lambda \in (0, 1/b)$

$$\begin{aligned} 0 \le \frac{I_3(\lambda)}{R(\lambda)} &= \int_M^\infty e^{-y} \frac{T(y/\lambda)}{T(1/\lambda)}\left(\frac{l(y/\lambda)}{l(1/\lambda)} - 1\right) dy \left(\int_0^\infty e^{-y} \frac{T(y/\lambda)}{T(1/\lambda)}\,dy\right)^{-1} \\ &\le c\frac{\int_M^\infty e^{-y} y^\varepsilon\,dy}{\int_1^\infty e^{-y}\,dy}. \end{aligned} \tag{5.1.18}$$

From relations (5.1.15)–(5.1.18) and Lemma 5.1.1 it follows that for some constant $c_1 > 0$

$$0 \le \limsup_{\lambda\downarrow 0}\left|\frac{I(\lambda)}{R(\lambda)}\right| \le c_1\left(\int_M^\infty e^{-y} y^\varepsilon\,dy + \limsup_{\lambda\downarrow 0}\frac{V(\delta/\lambda)}{V(1/\lambda)}\right). \tag{5.1.19}$$

If M tends to infinity and δ tends to zero in the right-hand side of (5.1.19), with the use of relation (5.1.1) we obtain

$$\limsup_{\lambda\downarrow 0}\left|\frac{I(\lambda)}{R(\lambda)}\right| = 0,$$

which implies (5.1.14). The lemma is proved. □

PROOF OF LEMMA 5.1.4. Relation (5.1.4) holds if and only if

$$I(\lambda) = \int_0^\infty ye^{-y}u(y/\lambda)\left(\frac{l(y/\lambda)}{l(1/\lambda)} - 1\right) dy = o(R(\lambda)) \tag{5.1.20}$$

as $\lambda \downarrow 0$, where

$$R(\lambda) = \int_0^\infty ye^{-y}u(y/\lambda)\,dy.$$

We represent the integral $I(\lambda)$ as the sum $I_1 + I_2 + I_3$ of the integrals over the intervals $[0, \delta)$, $[\delta, M]$, (M, ∞). By virtue of Lemma 3.5.9, there exists $\delta = \delta(\lambda) \to 0$ and $M = M(\lambda) \to \infty$ as $\lambda \downarrow 0$ such that

$$I_2 = \int_\delta^M ye^{-y}u(y/\lambda)\left(\frac{l(y/\lambda)}{l(1/\lambda)} - 1\right) dy = o(R(\lambda)). \tag{5.1.21}$$

Let us estimate the integral I_1:

$$\begin{aligned} 0 \ge I_1 &= \int_0^\delta ye^{-y}u(y/\lambda)\left(\frac{l(y/\lambda)}{l(1/\lambda)} - 1\right) dy \ge -\int_0^\delta ye^{-y}u(y/\lambda)\,dy \\ &= -\lambda^2\int_0^{\delta/\lambda} xu(x)e^{-\lambda x}\,dx \asymp -\lambda^2\int_0^{\delta/\lambda} xu(x)\,dx, \end{aligned} \tag{5.1.22}$$

as $\lambda \downarrow 0$. We observe that

$$\begin{aligned} R(\lambda) &= \int_0^\infty ye^{-y}u(y/\lambda)\,dy = \lambda^2\int_0^\infty xu(x)e^{-\lambda x}\,dx \\ &\ge \lambda^2\int_0^{1/\lambda} xu(x)e^{-\lambda x}\,dx \ge \lambda^2 e^{-1}\int_0^{1/\lambda} xu(x)\,dx. \end{aligned}$$

Therefore, in view of (5.1.22) and condition (5.1.3), as $\lambda \downarrow 0$

$$I_1 = o(R(\lambda)). \tag{5.1.23}$$

Further, for sufficiently small λ

$$\begin{aligned} 0 &\le \frac{I_3}{R(\lambda)} = \int_M^\infty ye^{-y}\frac{u(y/\lambda)}{u(1/\lambda)}\left(\frac{l(y/\lambda)}{l(1/\lambda)} - 1\right) dy \left(\int_0^\infty ye^{-y}\frac{u(y/\lambda)}{u(1/\lambda)}\,dy\right)^{-1} \\ &\le \frac{\int_M^\infty e^{-y}y^{1+\varepsilon}\,dy}{\int_1^\infty ye^{-y}\,dy}. \end{aligned} \tag{5.1.24}$$

Relation (5.1.20) follows from (5.1.22)–(5.1.24). The lemma is proved. □

PROOF OF LEMMA 5.1.5. Relation (5.1.5) holds if and only if

$$I(\lambda) = \int_0^\infty e^{-y} V(y/\lambda) \left(\frac{L(1/\lambda)}{L(y/\lambda)} - 1 \right) dy = o(R(\lambda)), \tag{5.1.25}$$

where

$$R(\lambda) = \int_0^\infty e^{-y} V(y/\lambda)\, dy.$$

As before, we represent the integral $I(\lambda)$ as the sum of the integrals $I_1(\lambda)$, $I_2(\lambda)$, $I_3(\lambda)$ over the intervals $[0, \delta)$, $[\delta, M]$, (M, ∞) respectively. By virtue of Lemma 3.5.9, there exist $\delta = \delta(\lambda)$ and $M = M(\lambda)$ such that, as $\lambda \downarrow 0$,

$$I_2 = o(R(\lambda)). \tag{5.1.26}$$

Further,

$$\begin{aligned} 0 \geq I_1 &= \int_0^\delta e^{-y} V(y/\lambda) \left(\frac{L(1/\lambda)}{L(y/\lambda)} - 1 \right) dy \\ &\geq -\int_0^\delta e^{-y} V(y/\lambda)\, dy \geq -\lambda \int_0^{\delta/\lambda} V(x)\, dx. \end{aligned}$$

In addition,

$$R(\lambda) = \int_0^\infty e^{-y} V(y/\lambda)\, dy \geq \int_0^1 e^{-y} V(y/\lambda)\, dy \geq e^{-1} \int_0^{1/\lambda} V(x)\, dx.$$

As $t \to \infty$, $\tau = o(t)$, we observe that

$$\int_0^\tau V(u)\, du \Big/ \int_0^t V(u)\, du \leq \int_0^\tau V(u)\, du \Big/ \left(\frac{t}{\tau} \int_0^\tau V(u)\, du \right) = \frac{\tau}{t} \to 0.$$

Therefore,

$$I_1 = o(R(\lambda)), \qquad \lambda \downarrow 0. \tag{5.1.27}$$

Since $V(t)$ is weakly oscillating, by virtue of Corollary 1.4.4 there exist constants c and σ such that for sufficiently small λ the inequalities

$$\begin{aligned} 0 \leq \frac{I_3}{R(\lambda)} &= \int_M^\infty e^{-y} \frac{V(y/\lambda)}{V(1/\lambda)} \left(\frac{L(1/\lambda)}{L(y/\lambda)} - 1 \right) dy \left(\int_0^\infty e^{-y} \frac{V(y/\lambda)}{V(1/\lambda)}\, dy \right)^{-1} \\ &\leq \frac{c \int_M^\infty e^{-y} y^\sigma\, dy}{\int_1^\infty e^{-y}\, dy} \end{aligned} \tag{5.1.28}$$

are true. Relation (5.1.25) follows from (5.1.26)–(5.1.28). The lemma is proved. □

5.2. The asymptotic behaviour of the kth record times

In this section, in the context of a particular record model where $\xi_n \equiv 1$, $n \in \mathbf{N}$, we deal with more general objects, the so-called kth record times. So, let us consider a sequence $\eta_0, \eta_1, \eta_2, \dots$ of independent random variables with one and the same continuous distribution function. For any $n \in \mathbf{Z}_+$, by the random variables $\eta_0, \eta_1, \dots, \eta_n$ we construct the set of order statistics

$$\eta_{0,n} \le \eta_{1,n} \le \dots \le \eta_{n,n}.$$

We define the *kth record times*, $\{\nu^{(k)}(n),\ n \in \mathbf{Z}_+\}$, $k \in \mathbf{N}$, as follows:

$$\nu^{(k)}(0) = k-1, \quad \nu^{(k)}(n+1) = \min\{j > \nu^{(k)}(n) \colon \eta_j > \eta_{j-k,j-1}\}, \quad n \in \mathbf{Z}_+.$$

To get the ordinary record times $\{\nu(n),\ n \in \mathbf{Z}_+\}$, it suffices to set $k = 1$. The kth record times are introduced in (Dziubdziela, 1977; Dziubdziela, Kopocinski, 1976). Their generating functions are obtained in (Nevzorov, 1990).

In this section we study the asymptotic behaviour of the probability that $\nu^{(k)}(n) > t$, where k and n are fixed, $t \to \infty$.

THEOREM 5.2.1. *For all $k, n \in \mathbf{N}$, as $t \to \infty$*

$$\mathsf{P}\{\nu^{(k)}(n) > t\} \sim \frac{k^n (k-1)!}{(n-1)!} t^{-k} (\ln t)^{n-1}. \tag{5.2.1}$$

This theorem is proved in (Yakymiv, 1995).

PROOF. We introduce the random variables

$$\rho^{(k)}(n) = \nu^{(k)}(n-1) + 1, \qquad k, n \in \mathbf{N}.$$

In (Nevzorov, 1990), it is shown that for $k, n \in \mathbf{N}$, $s \in [0, 1)$

$$\mathsf{E} s^{\rho^{(k)}(n)} = r_{kn}(-\ln(1-s)), \tag{5.2.2}$$

where

$$r_{kn}(t) = \frac{k^n}{(n-1)!} \int_0^t x^{n-1} e^{-kx} (1 - e^{-(t-x)})^{k-1}\, dx, \qquad t \ge 0. \tag{5.2.3}$$

We set $c_{kn} = k^n/(n-1)!$, $m = k-1$. Let us calculate $r_{kn}(t)$. Since

$$(1 - e^{-(t-x)})^{k-1} = \sum_{i=0}^{m} (-1)^i \binom{i}{m} e^{-i(t-x)},$$

from (5.2.3) we obtain

$$\begin{aligned} r_{kn}(t) &= c_{kn} \int_0^t x^{n-1} e^{-kx} \sum_{i=0}^{m} (-1)^i \binom{i}{m} e^{-i(t-x)}\, dx \\ &= c_{kn} \sum_{i=0}^{m} (-1)^i \binom{i}{m} e^{-it} \int_0^t x^{n-1} e^{-(k-i)x}\, dx. \end{aligned} \tag{5.2.4}$$

Let $j = k - i$, $y = (k-i)x$. We observe that

$$\int_0^t x^{n-1} e^{-(k-i)x}\,dx = j^{-n} \int_0^{tj} y^{n-1} e^{-y}\,dy$$
$$= j^{-n}(n-1)! \left(1 - e^{-jt} \sum_{l=0}^{n-1} \frac{(jt)^l}{l!}\right). \qquad (5.2.5)$$

From (5.2.4) and (5.2.5) it follows that

$$r_{kn}(t) = c_{kn} \sum_{i=0}^{m} (-1)^i \binom{i}{m} e^{-it} j^{-n}(n-1)! \left(1 - e^{-jt} \sum_{l=0}^{n-1} \frac{j^l t^l}{l!}\right)$$
$$= k^n \left(\sum_{i=0}^{m} (-1)^i \binom{i}{m} e^{-it} j^{-n} - e^{-kt} \sum_{i=0}^{m} \sum_{l=0}^{n-1} (-1)^i \binom{i}{m} \frac{j^{l-n} t^l}{l!}\right),$$

hence we obtain

$$r_{kn}(t) = k^n \left(\sum_{i=0}^{m} (-1)^i \binom{i}{m} e^{-it} j^{-n} - e^{-kt} \sum_{l=0}^{n-1} a_l t^l\right), \qquad (5.2.6)$$

where

$$a_l = \sum_{i=0}^{m} (-1)^i \binom{i}{m} \frac{(k-i)^{l-n}}{l!}, \qquad l = 0, \ldots, n-1. \qquad (5.2.7)$$

For $s \in [0, 1)$, let

$$h_{kn}(s) = \sum_{i=1}^{\infty} \mathbf{P}\{\rho^{(k)}(n) > i\} s^i. \qquad (5.2.8)$$

By (5.2.2), (5.2.8), for $s \in [0, 1)$ we see that

$$h_{kn}(s) = \frac{1 - r_{kn}(-\ln(1-s))}{(1-s)}. \qquad (5.2.9)$$

From (5.2.6) and (5.2.9) we arrive at

$$h_{kn}(s) = k^n \left(-\sum_{i=0}^{m} (-1)^i \binom{i}{m} (1-s)^{i-1} (k-i)^{-n} + (1-s)^m \sum_{l=0}^{n-1} a_l (-\ln(1-s))^l\right).$$

By differentiating the last relation m times with respect to s we obtain

$$h_{kn}^{(m)}(s) = k^n \sum_{l=0}^{n-1} \sum_{j=0}^{m} a_l \frac{d^j}{ds^j} (1-s)^m \frac{d^{m-j}}{ds^{m-j}} (-\ln(1-s))^l \binom{j}{m}. \qquad (5.2.10)$$

We set $m^{[j]} = m(m-1)\cdots(m-j+1)$, $m^{[0]} = 1$, and obtain

$$\frac{d^j}{ds^j}(1-s)^m = m^{[j]}(1-s)^{m-j}(-1)^j. \tag{5.2.11}$$

For $r \in \mathbf{N}$ we observe that

$$\frac{d^r}{ds^r}(-\ln(1-s))^l = \sum_{i=1}^{r} b_{i\,r}\, l^{[i]} \frac{(-\ln(1-s))^{l-i}}{(1-s)^r}, \tag{5.2.12}$$

where the generating function

$$g_r(s) = \sum_{i=1}^{r} b_{i,r} s^i$$

is of the form

$$g_r(s) = s(s+1)\cdots(s+r-1). \tag{5.2.13}$$

For $r = 1$, (5.2.12), (5.2.13) hold indeed, because

$$\frac{d}{ds}(-\ln(1-s))^l = l\,\frac{(-\ln(1-s))^{l-1}}{(1-s)}, \quad b_{1,1} = 1, \quad g_1(s) = s.$$

If (5.2.12), (5.2.13) hold for some $r \in \mathbf{N}$, then

$$\begin{aligned}\frac{d^{r+1}}{ds^{r+1}}(-\ln(1-s))^l &= \sum_{i=1}^{r} b_{i,r}\, l^{[i]} r \frac{(-\ln(1-s))^{l-i}}{(1-s)^{r+1}} + \sum_{i=1}^{r} b_{i,r}\, l^{i+1} \frac{(-\ln(1-s))^{l-i-1}}{(1-s)^{r+1}} \\ &= \Sigma_1 + \Sigma_2.\end{aligned}$$

We observe that Σ_2 admits the representation

$$\Sigma_2 = \sum_{i=2}^{r=1} b_{i-1,r}\, l^{[i]} \frac{(-\ln(1-s))^{l-i}}{(1-s)^{r+1}}.$$

Therefore,

$$\Sigma_1 + \Sigma_2 = \sum_{i=1}^{r+1} (r b_{i,r} + b_{i-1,r})\, l^{[i]} \frac{(-\ln(1-s))^{l-i}}{(1-s)^{r+1}},$$

where $b_{0,r} = b_{1,r} = 0$. Thus,

$$\frac{d^{r+1}}{ds^{r+1}}(-\ln(1-s))^l = \sum_{i=1}^{r+1} b_{i,r+1}\, l^{[i]} \frac{(-\ln(1-s))^{l-i}}{(1-s)^{r+1}},$$

where $b_{i,r+1} = r b_{i,r} + b_{i-1,r}$, $i = 1, \ldots, r+1$. From the last two relations it follows that

$$\begin{aligned}g_{r+1}(s) &= \sum_{i=1}^{r+1} (r b_{i,r} + b_{i-1,r}) s^i = r \sum_{i=1}^{r+1} b_{i,r} s^i + \sum_{j=0}^{r} b_{j,r} s^{j+1} \\ &= r g_r(s) + s g_r(s) = (r+s) g_r(s).\end{aligned}$$

Taking into account (5.2.13), we arrive at the equality

$$g_{r+1}(s) = s(s+1)\cdots(s+r).$$

Thus, relations (5.2.12) and (5.2.13) are proved. From (5.2.10)–(5.2.12) it follows that

$$h_{kn}^{(m)}(s) = k^n \sum_{l=0}^{n-1} \sum_{j=0}^{m} (-1)^j a_l m^{[j]} \binom{j}{m} \times \left(\sum_{i=1}^{m-j} b_{i,r} l^{[i]} (-\ln(1-s))^{l-i} + (-\ln(1-s))^l \chi_r \right),$$

where

$$r = m - j, \qquad \chi_r = \begin{cases} 1, & r = 0, \\ 0, & r \neq 0. \end{cases}$$

Therefore,

$$h_{kn}^{(m)}(s) = k^n \sum_{l=0}^{n-1} \sum_{j=0}^{m} (-1)^j a_l m^{[j]} \binom{j}{m} \left(\sum_{i=0}^{m-j} B_{i,r} l^{[i]} (-\ln(1-s))^{l-i} \right),$$

where $B_{i,r} = b_{i,r}$ for $i \in \mathbf{N}$ and $B_{0,r} = \chi_r$. By differentiating the last equality with respect to s we obtain

$$h_{kn}^{(k)}(s) = k^n \sum_{l=0}^{n-1} \sum_{j=0}^{m} (-1)^j a_l m^{[j]} \binom{j}{m} \left(\sum_{i=0}^{m-j} B_{i,r} l^{[i+1]} \frac{(-\ln(1-s))^{l-i-1}}{(1-s)} \right).$$

Here the greatest power $n-2$ of logarithm occurs for $l = n-1$, $i = 0$, $j = m$. Thus, for $n > 1$, as $s \uparrow 1$,

$$\frac{d^k}{ds^k} h_{kn}(s) \sim k^n (-1)^m a_{n-1} m!\,(n-1) \frac{(-\ln(1-s))^{n-2}}{(1-s)} \tag{5.2.14}$$

(below we will show that $a_{n-1} \neq 0$). Let us calculate a_{n-1}. By (5.2.7),

$$a_{n-1} = \frac{1}{(n-1)!} \sum_{i=0}^{m} (-1)^i \binom{i}{m} (k-i)^{-1}.$$

Setting $j = m - i$ in the last sum and recalling that $\binom{j}{m} = \binom{i}{m}$, $k = m+1$, we obtain

$$a_{n-1} = \frac{(-1)^m}{(n-1)!} \sum_{j=0}^{m} (-1)^j \binom{j}{m} (j+1)^{-1} = \frac{(-1)^m}{(n-1)!} A, \tag{5.2.15}$$

where

$$A = \sum_{j=0}^{m} (-1)^j \binom{j}{m} (j+1)^{-1} = -\sum_{t=1}^{k} (-1)^j \binom{t-1}{m} t^{-1}. \tag{5.2.16}$$

We introduce the generating function

$$B(s) = \sum_{t=1}^{k} s^t \binom{t-1}{m} t^{-1} \tag{5.2.17}$$

and observe that

$$B'(s) = \sum_{t=1}^{k} s^{t-1} \binom{t-1}{m} = \sum_{j=0}^{m} s^j \binom{j}{m} = (1+s)^m.$$

Therefore,

$$B(s) = \int_0^s (1+u)^m \, du = \frac{(1+u)^k}{k}\bigg|_0^s = \frac{(1+s)^k - 1}{k}. \tag{5.2.18}$$

By virtue of (5.2.16), (5.2.17), and (5.2.18), $A = -B(-1) = 1/k$. In view of (5.2.15), hence it follows that

$$a_{n-1} = (-1)^m/((n-1)!k).$$

Therefore, (5.2.14) can be rewritten as follows: as $s \uparrow 1$,

$$\frac{d^k}{ds^k} h_{kn}(s) \sim \frac{k^{n-1}}{(n-2)!}(k-1)! \frac{(-\ln(1-s))^{n-2}}{(1-s)} \tag{5.2.19}$$

Now from (5.2.19) and Theorem 1.6.4 we arrive at

$$\mathsf{P}\{\rho^{(k)}(n) > t\} \sim \frac{k^{n-1}(k-1)!}{(n-2)!} t^{-k} (\ln t)^{n-2}$$

for $t \to \infty$ and fixed $k, n \in \mathbf{N}$, $n > 1$. The theorem is proved. □

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